Sequential Point Process Model and Bayesian Inference for Spatial Point Patterns with Linear Structures

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Outline

Data

- Dataset 1: Barrows
- Dataset 2: Mountain tops

Model

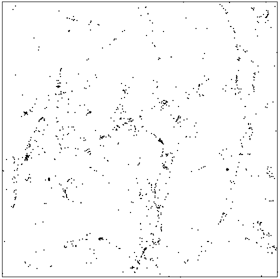
- Model construction
- Simulation algorithm

Inference

- Bayesian model: likelihood and priors
- MCMC based parameter estimation

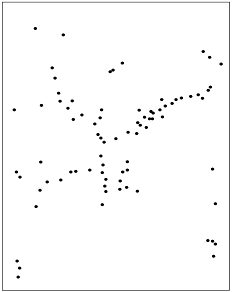
Dataset 1: Barrows

- A barrow is a bronze age burial site resembling a small hill.
- These are important sources of information for archaologists.
- They are often placed roughly in linear structures.



Dataset 2: Mountain tops

- Mountains ridges means that "local" tops are often forming linear structures.



Linear structures

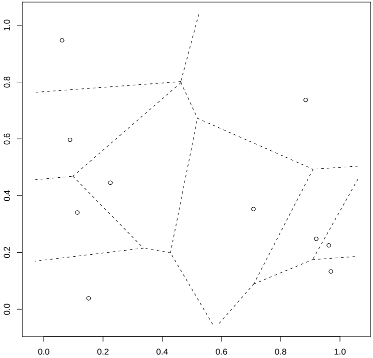
- In this talk we will consider a model capable of generating linear formations.
- Roughly speaking, this model generates linear structures by moving points closer to other points.
- Interpretation of the model:
  - Barrows: Here the model is interpreted as dead people are buried close to previously buried people.
  - Mountains: No reasonable interpretation - the model should not be thought of as representing actual mechanics.

Outline

Model construction

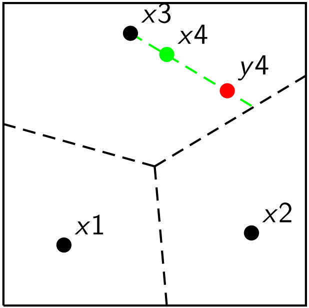
- Point process x defined on window W.
- $x = x_c \cup x_b$ with n points.
- Number of points in x_c, k, is binom(n, q).
- Background process:
  - x_b consists of i.i.d. uniformly distributed points on W
- Cluster process:
  - Sequential construction.
  - A point is initially uniformly distributed independently of everything else.
  - With probability p this point is moved closer to the closest previous point; otherwise it keeps its original position.

Voronoi tesselations



- Voronoi tesselation: an area is associated with the closest point in the point process

Moving points



- Density for new position:

$$h(x_i|\{x_1,\ldots,x_{i-1}\};\sigma^2) \propto \exp(-r_i^2/(2\sigma^2)), \qquad 0 < r_i < l_i$$

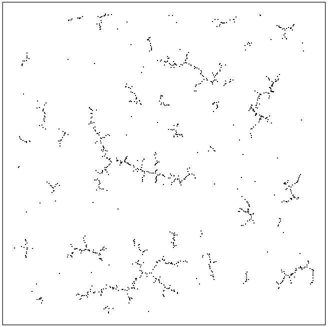
- Other distributions have also been tried.

Simulation algorithm

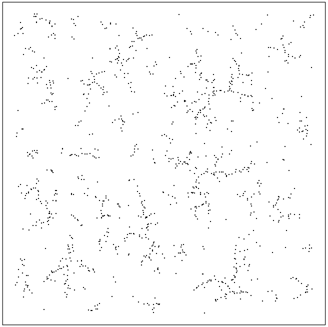
Fix number of points, and for each point do the following:

1. Find type of point i (cluster with prob. q, background otherwise)
2. If background point:
   2.1 Find coordinates - uniformly distributed on W
3. If cluster point:
   3.1 Find initial coordinates - uniformly distributed on W
   3.2 Move with probability p, otherwise keep position
   3.3 If move, find closest cluster point and move new point closer to this using Exp-distribution

Two simulations



$q = 0.95, p = 0.95, \sigma = 70$



$q = 0.90, p = 0.90, \sigma = 130$

Remember:

- q is the probability that a point is a cluster point (i.e. belongs to a linear structure)
- p is the probability is that a cluster point is moved (i.e. continues an existing linear structure)
- σ governs how close points in lines are located to each other

Outline

Likelihood and priors

- Let z be the observed point pattern x including type (cluster/background) and order of points.
- Likelihood:

$$L(q, p, \sigma^2 | z) = \binom{n}{k} q^k \left(\frac{1-q}{|W|} \right)^{n-k} \prod_{i=1}^{k} f(x_i | x_1, \ldots, x_{i-1}; p, \sigma^2)$$

  where

$$f(\cdot | x_1, \ldots, x_{i-1}; p, \sigma^2) = p \times h(\cdot | \{x_1, \ldots, x_{i-1}\}; \sigma^2) + (1-p) \times \frac{1}{|W|}$$

- Priors:
  - Independent priors for p, q, σ.
  - p, q: Uniform on $[0, 1]$.
  - σ: Flat inverse gamma or (improper) uniform on $[0, \infty)$.

Estimation of parameters

- Ideally we would find mean/maximum of the posterior

$$p(q, p, \sigma^2 | x) \propto g_1(p) g_2(q) g_3(\sigma^2) L(q, p, \sigma^2 | x)$$

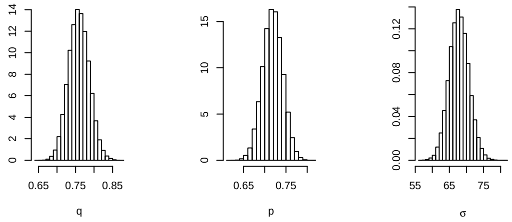
  but we only have closed form expression for $L(q, p, \sigma^2 | z)$, not $L(q, p, \sigma^2 | x)$.
- So we have a missing data problem:
  - The order of $x_c = \{x_1, \ldots, x_k\}$ is unknown.
  - Also it is unknown whether a point belongs to x_c or x_b.
- So we need to approximate the estimates of p, q, σ and the missing data by Markov chain Monte Carlo.

MCMC

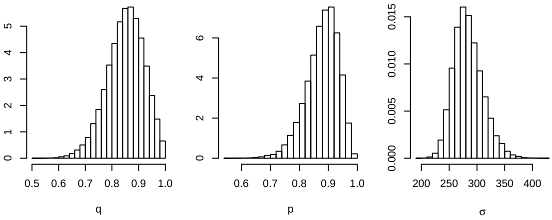
- We use Metropolis within Gibbs to obtain posterior.
- Updates:
  - A background point becomes a cluster point.
  - A cluster point becomes a background point.
  - Shifting the ordering of two succeeding cluster points.
  - Parameters p, q and σ^2: Metropolis update, normal proposal.
- Hastings ratios are easily obtained.
- Calculation times are not too bad.
- Mixing is fair.

Posterior distributions - parameters

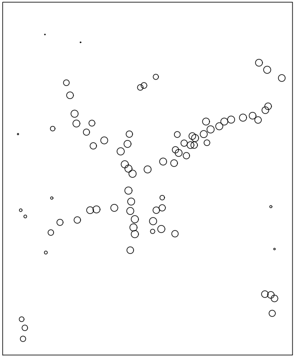
Barrows:

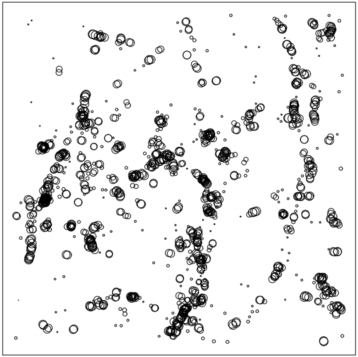


Mountains:



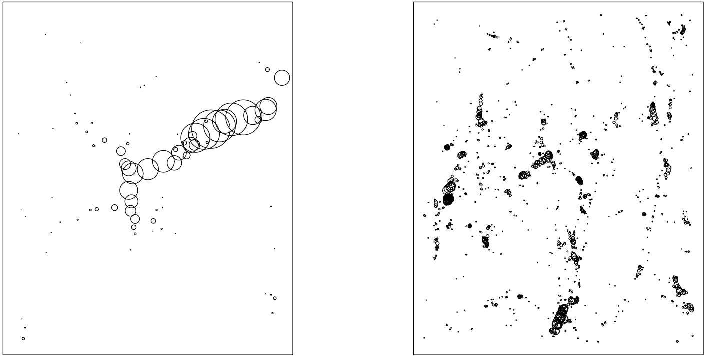
Posterior distributions - missing data





Circle radius indicates marginal posterior probability of a point being a cluster point.

Posterior distributions - missing data



Circle radius indicates the order in which the cluster points occur.

Concluding remarks

- Summing up: a new model with linear structures and MCMC-based Bayesian inference
- Model checking skipped in this talk
- Many extensions/modifications possible, e.g. inclusion of covariates, initial placements or moving mechanims

Thank you for your attention :-)