Planar Markov fields

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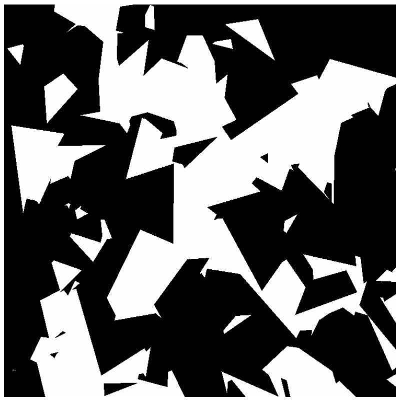
Arak–Surgailis polygonal Markov fields

form a coloured mosaic by

- isotropic Poisson line process skeleton for drawing polygonal contours;
- no line can be used more than once; many mosaics can be drawn on the same skeleton;
- adjacent polygons have different colours;
- probability distribution defined by Hamiltonian chosen so that
  - basic properties of the Poisson line process carry over,
  - dynamic representation in terms of particle system holds.



Background and aim

In (9), Schreiber and I introduced a class of binary random fields that can be understood as discrete versions of the two-colour Arak process.

Goal

extend the construction to

- allow for an arbitrary number of colours;
- relax the assumption that no polygons of the same colour can be joined by corners only;
- have a dynamical representation that can be used for sampling;
- satisfy a spatial Markov property.

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Regular linear tessellations

are countable families $\mathcal{T}$ of straight lines in $\mathbb{R}^2$ such that

- no three lines of $\mathcal{T}$ intersect at one point;
- a bounded subset of $\mathbb{R}^2$ can be hit by at most a finite number of lines from $\mathcal{T}$.

Fixed activity parameters $\pi_l \in (0, 1)$ are ascribed to each $l \in \mathcal{T}$.

Examples

- Poisson line process;
- regular planar lattice.

Polygonal configuration

- $D \subset \mathbb{R}^2$ bounded, open and convex;
- ∂D contains no nodes, that is, no intersections of two lines of $\mathcal{T}$;
- for all $l \in \mathcal{T}$, $\mathrm{card}(l \cap \partial D) = 2$;
- $J = \{1, \ldots, k\}$, $k \geq 2$.

$\hat{\Gamma}_D(\mathcal{T})$ is the set of all planar graphs γ in $D \cup \partial D$ with faces coloured by labels in J such that

- all edges of γ lie on the lines of $\mathcal{T}$;
- all vertices of γ in D are of degree 2, 3, or 4;
- all vertices of γ on ∂D, are of degree 1;
- no adjacent regions share the same colour.

$\Gamma_D(\mathcal{T})$ consists of all planar graphs γ in $\overline{D} = D \cup \partial D$ arising as interfaces between differently coloured regions in $\hat{\gamma} \in \hat{\Gamma}_D(\mathcal{T})$.

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Discrete polygonal field

$\hat{\mathcal{A}}_{\mathcal{H}_D}$ with **Hamiltonian** $\mathcal{H}_D : \hat{\Gamma}_D(\mathcal{T}) \mapsto \mathbb{R} \cup \{+\infty\}$ is defined by

$$\mathbb{P}\left(\hat{\mathcal{A}}_{\mathcal{H}_D} = \hat{\gamma}\right) = \frac{\exp\left[-\mathcal{H}_D(\hat{\gamma})\right] \prod_{e \in E^*(\gamma)} \pi_{l[e]}}{\mathcal{Z}[\mathcal{H}_D]},$$

where $\mathcal{Z}[\mathcal{H}_D]$ is the partition function, E^* denotes **primary edges**, i.e. maximal open connected collinear line segments consisting of multiple edges (due to T- or X-nodes), and $l[e] \in \mathcal{T}$ is the line containing e.

For a special choice of $\mathcal{H}$, the model has remarkable properties. Fix $k \geq 2$, $\alpha_V \in [0,1]$. Set $\alpha_X = 1 - \alpha_V$ and

$$\alpha_T = \frac{1}{2}\left(1 - \frac{k-2}{k-1}\alpha_X\right); \qquad \epsilon = \frac{\alpha_V}{k-1} + \frac{k-2}{k-1}\alpha_T.$$

Consistent polygonal fields

Define Hamiltonian $\Phi_D(\hat{\gamma})$ by

$$\begin{aligned} = &\quad -N_V(\gamma)\log\alpha_V - N_T(\gamma)\log((k-1)\alpha_T) - N_X(\gamma)\log((k-1)\alpha_X) \\ + &\quad \mathrm{card}(E(\gamma))\log(k-1) \\ - &\quad \sum_{e\in E(\gamma)}\sum_{l\in\mathcal{T},\, l\not\sim e}\log(1-\epsilon\pi_l) + \sum_{n(l_1,l_2)\in\gamma}\log\left(1-\frac{\alpha_V}{k-1}\pi_{l_1}\pi_{l_2}\right) \end{aligned}$$

where $N(V)$, $N(T)$, $N(X)$ are the number of V-, T-, and X-nodes,

$$\mathcal{Z}[\Phi_D] = k\prod_{l\in\mathcal{T},\, l\cap D\neq\emptyset}(1+\pi_l)\prod_{n(l_1,l_2)\in D}\left(1-\frac{\alpha_V}{k-1}\pi_{l_1}\pi_{l_2}\right)^{-1}.$$

Theorem

$\hat{\mathcal{A}}_{\Phi_D}\cap D' =^d \hat{\mathcal{A}}_{\Phi_{D'}}$, for bounded open convex $D'\subseteq D$.

Proof: Dynamic representation

Interpret

$$(t, y) \in D$$

as: y is the 1D position of a particle at time t.

W.l.o.g. assume no line of $\mathcal{T}$ is parallel to the spatial axis.

Birth sites

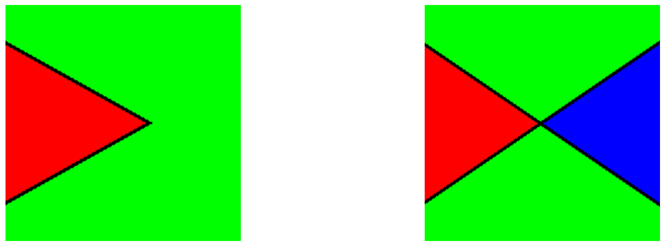
- at each node $n(l_1, l_2) \in \mathcal{T} \cap D$ w.p. $\alpha_V \pi_{l_1} \pi_{l_2}/(k-1)$ two particles are born, moving forward in time along l_1 and l_2 **unless** some previously born particle hits the node;
- at each entry point $\mathrm{in(l, D)}$ of lines $l \in \mathcal{T}$ into D, w.p. $\pi_l/(1+\pi_l)$ a single particle is born, moving forward in time along l.

Colours are chosen uniformly conditional on not clashing with the colour just prior (left) to the birth.

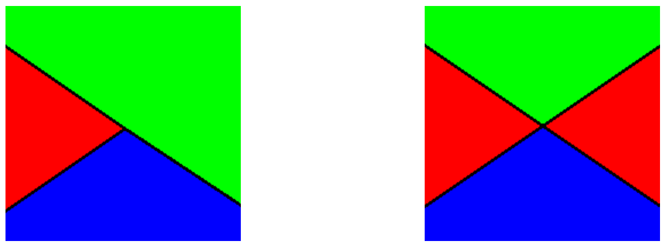
Dynamic representation: Collisions

of two particles at some moment t with $(t, y) = n(l_i, l_j) \in D$

a if the colours above and below are identical, w.p. α_V both particles die, w.p. α_X both survive and a new colour is chosen w.p. $1/(k-1)$ for each admissible colour;



b if the colours above and below clash, w.p. α_T, each of the two particles survives while the other dies; w.p. $1 - 2\alpha_T$, both survive and a new colour is chosen w.p. $1/(k-2)$ for each admissible colour.



Note: a collision prevents a birth at that node.

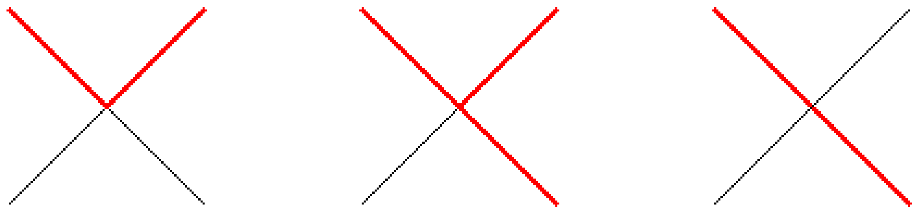
Dynamic representation: Updates at nodes

Whenever a particle moving along $l_i \in \mathcal{T}$ reaches $n(l_i, l_j)$, it

a will change velocity to continue along l_j w.p. $\alpha_V \pi_{l_j}/(k-1)$;

b splits into two particles moving along l_i and l_j w.p. $(k-2)\alpha_T \pi_{l_j}/(k-1)$; a new colour is chosen uniformly from the $k-2$ possibilities;

c keeps moving along l_i otherwise (w.p. $1 - \epsilon\pi_{l_j}$).



These dynamics define a random coloured polygonal configuration that can be shown to coincide in distribution with $\hat{A}_{\Phi_D}$. Consistency follows.

Further properties

For $\hat{\mathcal{A}}_{\Phi_D}$, the following hold.

Linear sections: For any line l containing no nodes of $\mathcal{T}$, $\hat{\mathcal{A}}_{\Phi_D} \cap l$ coincides in law with $\Lambda_{\mathcal{T}} \cap l$, where each $l^* \in \mathcal{T}$ belongs to $\Lambda_{\mathcal{T}}$ w.p. $\frac{\pi_{l^*}}{1+\pi_{l^*}}$ independently of others.

Spatial Markov: For a piecewise smooth simple closed curve $\theta \subset \mathbb{R}^2$ containing no nodes of $\mathcal{T}$, the conditional distribution in the interior of θ depends on the exterior configuration only through the intersection points and intersection directions of θ with the edges of the polygonal field and through the colouring of the field along θ.

Notes

- properties resemble those of continuous Arak–Surgailis fields;
- the spatial Markov property implies the local Markov property.

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Examples

$\mathcal{T}$ consists of tilted line bundles with density 0.07 on $[-128, 128]^2$ and $\pi_l \equiv 1/2$ for all $l \in \mathcal{T}$.

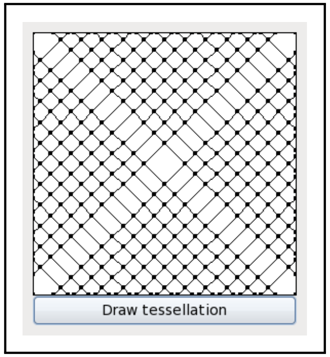

Figure 1: Tilted line bundle.

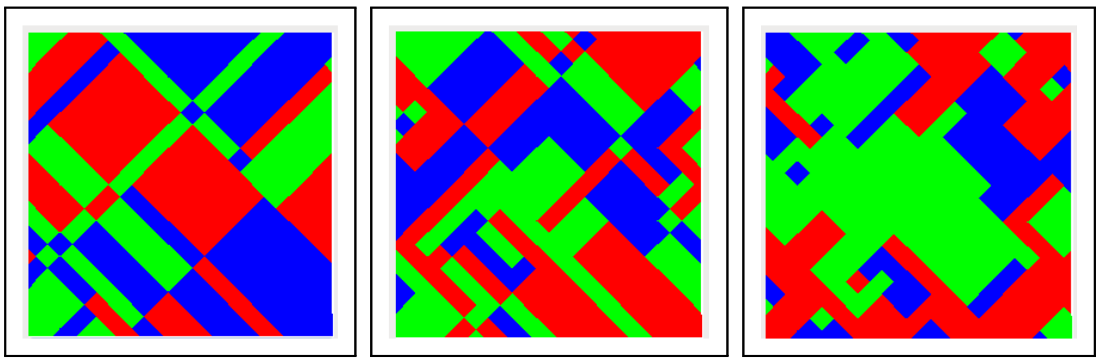

Figure 2: Samples of $\hat{A}_{\Phi_D}$ with $k = 3$, $\alpha_V = 0$ (left), $\alpha_V = 1/2$ (centre) and $\alpha_V = 1$ (right).

Birth–death dynamics for consistent fields

For $k = 2$, $\alpha_V = 1$ (Schreiber and Van Lieshout, 2010)

- for each admissible γ, there are only two colourings;
- all particles die upon collision.

For $k > 2$, these facts no longer apply.

For $k > 2$, use continuous time dynamics with three types of updates:

- add a particle birth;
- delete a (discarded) particle birth at rate 1;
- recolour the graph by Knuth shuffling at rate $\tau > 0$.

To find the birth rate, solve the balance equations to obtain rate

$$c\pi_{l_1}\pi_{l_2}/(1 - c\pi_{l_1}\pi_{l_2})$$

with $c = \alpha_V/(k-1)$ for vacant internal node $n(l_1, l_2)$. If $n(l_1, l_2)$ is hit by some previously born particle, the birth is discarded. The boundary birth rate at $\mathrm{in(l, D)}$ is π_l.

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Birth–death dynamics for consistent fields (ctd)

The trajectories of particle(s) emitted at a birth update and their collisions are chosen in accordance with the dynamic representation, re-using existing trajectories whenever possible. A dual reasoning is applied to deaths.

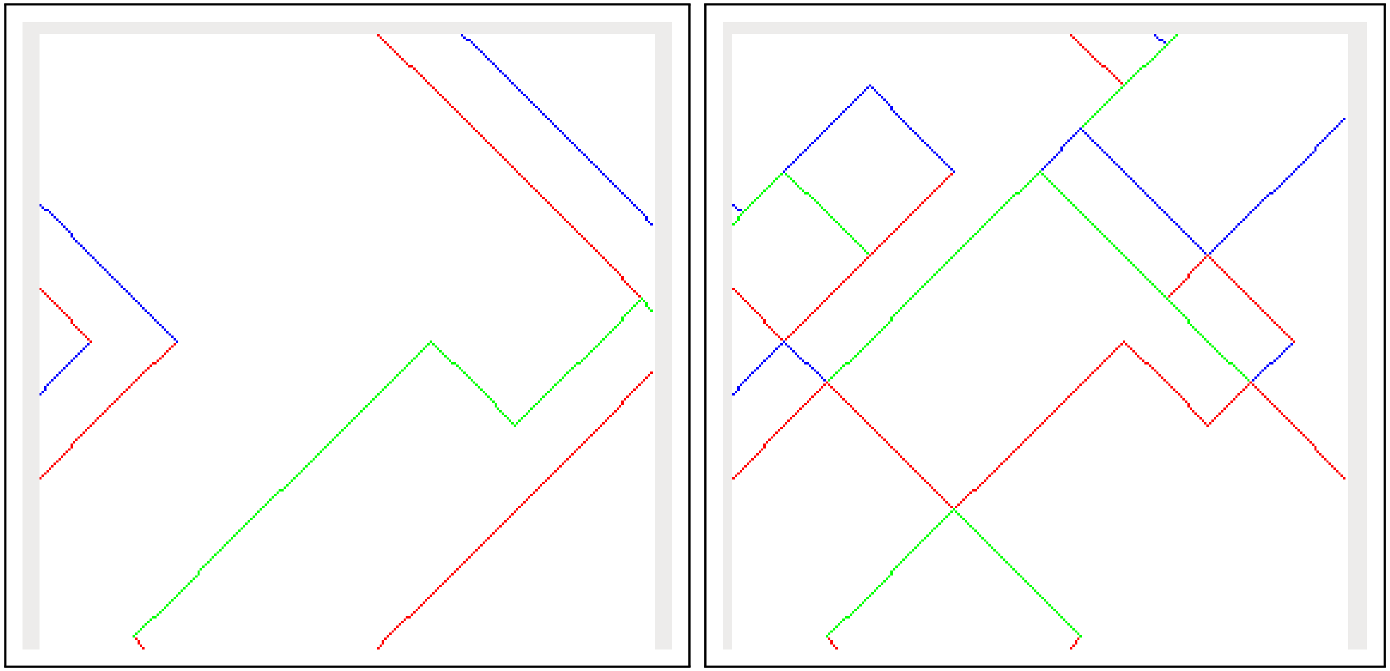

Figure 3: Boundary birth update: Old configuration (left), new configuration (right). Line colour corresponds to label below the line.

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Accept-reject sampler

for $\hat{\mathcal{A}}_{\Phi_D+\mathcal{H}}$ accepts a new state $\hat{\gamma}'$ with probability

$$\min\left(1, \exp\left[\mathcal{H}(\hat{\gamma}) - \mathcal{H}(\hat{\gamma}')\right]\right).$$

Example

$$\mathcal{H}(\hat{\gamma}) = \beta\left[-\sum_{e\in E(\gamma)}\sum_{l\in\mathcal{T},\, l\not\sim e}\log(1-\epsilon\pi_l)\right]$$

For $\beta > 0$, there tend to be more large, fat cells; for $\beta < 0$ more thin, elongated shapes.

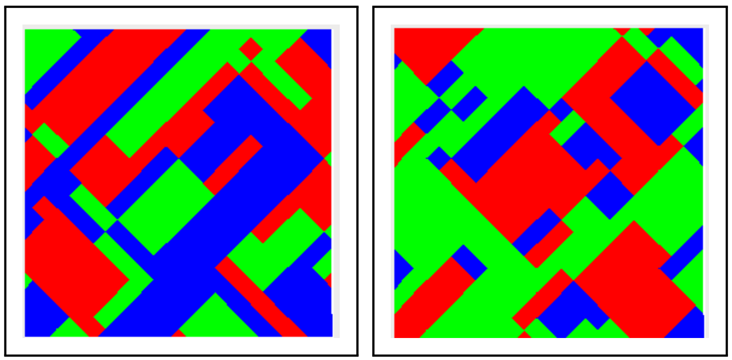

Figure 4: Samples of $\hat{\mathcal{A}}_{\Phi_D+\mathcal{H}}$ with $\alpha_V = 1/2$ and $\beta = -1/4$ (left) and $\beta = 1/4$ (right) for $\tau = 10$ and time $15,000$ (over five million updates).

Summary

We presented a class of multi-colour discrete random fields on finite graphs

- inspired by consistent polygonal Markov fields;
- that have an explicit partition function;
- that generalise the binary fields considered before;
- cover classic Gibbs fields;
- and have a dynamic representation on which new simulation algorithms may be based.

- In contrast to the continuum, collinear edges are allowed.
- The fixed regular linear tessellation may be replaced by a random one (e.g. Poisson line process) or be determined by data (image segmentation).

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