Semi-parametric estimation of the Poisson intensity parameter for stationary Gibbs point processes

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Stationary Gibbs point processes on $\mathbb{R}^d$

- We define a point process X in $\mathbb{R}^d$ as a locally finite random subset of $\mathbb{R}^d$, i.e. $N(\Lambda) = n(X_\Lambda)$ is a finite random variable whenever $\Lambda \subset \mathbb{R}^d$ is a bounded region.
- If the distribution of X is translation invariant, we say that X is stationary.
- we are interested in stationary Gibbs point processes on $\mathbb{R}^d$ which may be defined through of Papangelou conditional intensity $\lambda : \mathbb{R}^d \times N_{lf} \longrightarrow \mathbb{R}^+$.
- The Papangelou conditional intensity has the interpretation that $\lambda(u, x)du$ as the probability that the process X has to send a point in a region of du around a point u which also respects the existing configuration outside of the du.

Examples Gibbs point processes

- **Strauss point process:**

$$\lambda(u,x) = \beta\gamma^{n_{[0,R]}(u,x)}$$

where $\beta > 0, \gamma \in [0,1]$, $n_{[0,R]}(u,x) = \sum_{v \in x_\Lambda} \mathbf{1}(\|v-u\| \le R)$.

- **Strauss point process with Hard-Core:**
  - If all points are at distance greater than δ from each other

$$\lambda(u,x) = \beta\gamma^{n_{[0,R]}(u,x)}.$$

  - otherwise $\quad \lambda(u,x) = 0$.

- **Piecewise Strauss point process:**

$$\lambda(u,x) = \beta \prod_{j=1}^{p} \gamma_j^{n_{[R_{j-1},R_j]}(u,x)}$$

where $n_{[R_{j-1},R_j]}(u,x) = \sum_{v \in x_\Lambda} \mathbf{1}(\|v-u\| \in [R_{j-1}, R_j])$ where $R_0 = 0 < R_1 < \ldots < R_p < +\infty$.

Position of the problem

We consider Gibbs models such that the Papangelou conditional intensity can be written for $u \in \mathbb{R}^d$ and $x \in N_{lf}$

$$\lambda(u, x; \beta^\star) = \beta^\star \, \widetilde{\lambda}(u, x),$$

where

- $\beta^\star$ is the "Poisson intensity" parameter.
- $\widetilde{\lambda}$ is a function from $\mathbb{R}^d \times N_{lf}$ to $\mathbb{R}^+$.
- **[FR]** The Papangelou conditional intensity satisfies

  $$\lambda(u, x; \beta^\star) = \lambda(u, x_{B(u,R)}; \beta^\star), \quad \text{for any } u \in \mathbb{R}^d, x \in N_{lf}$$

  and such that $\widetilde{\lambda}(u, \emptyset) = 1$.

  We propose to estimate $\beta^\star$ independently of $\widetilde{\lambda}$ based on a single observation of a stationary Gibbs point process in $\mathbb{R}^d$, denoted X, in a domain $\Lambda_n \bigoplus \widetilde{R}$, where $(\Lambda_n)_{n \geq 0}$ is a sequence of increasing cubes and $\widetilde{R} \geq R$ and we do not assume R known, but only know an upper bound $\widetilde{R}$.

Definition of the estimator

- For all nonnegative measurable functions h on $\mathbb{R}^d \times N_{lf}$, then

$$\mathbf{E}\left[\sum_{u \in X} h(u, X \backslash u)\right] = \mathbf{E}\left[\int_{\mathbb{R}^d} h(u, X)\lambda(u, X; \beta^\star) du\right] \qquad (1)$$

  (where the left hand side is finite if and only if the right hand side is finite).
- With the choice of h defined by

$$\textbf{[CH]} : h(u, X) = \mathbf{1}\left(X \cap B(u, \widetilde{R}) = \emptyset\right).$$

$$
\begin{aligned}
\mathbf{E}\left[\sum_{u\in X_{\Lambda_n}} h(u,X\backslash u)\right] &= \mathbf{E}\left[\overbrace{\sum_{u\in X_{\Lambda_n}} \mathbf{1}((X\backslash u)\cap B(u,\widetilde{R})=\emptyset)}^{N_{\Lambda_n}(X;\widetilde{R})}\right] \\
&= \beta^\star \mathbf{E}\left[\int_{\Lambda_n} \mathbf{1}\left(X\cap B(u,\widetilde{R})=\emptyset\right)\widetilde{\lambda}(u,X)du\right] \\
&= \beta^\star \mathbf{E}\left[\int_{\Lambda_n} \mathbf{1}(X\cap B(u,\widetilde{R})=\emptyset)\widetilde{\lambda}(u,X_{B(u,R)})du\right] \\
&= \beta^\star \mathbf{E}\left[\int_{\{u,X\cap B(u,\widetilde{R})=\emptyset\}} \widetilde{\lambda}(u,\emptyset)du\right] \\
&= \beta^\star \mathbf{E}\left[\underbrace{\int_{\Lambda_n} \mathbf{1}(X\cap B(u,\widetilde{R})=\emptyset)du}_{V_{\Lambda_n}(X;\widetilde{R})}\right].
\end{aligned}
$$

Consistency of the estimator

With the ergodic theorem suggests to estimate $\beta^\star$ by the estimator

$$\widehat{\beta}_n(X;\widetilde{R}) = \frac{|\Lambda_n|^{-1} N_{\Lambda_n}(X;\widetilde{R})}{|\Lambda_n|^{-1} V_{\Lambda_n}(X;\widetilde{R})}.$$

Proposition

Let X be stationary Gibbs point process, under the assumptions **[FR]** *and* **[CH]***. Then for any fixed $0 < R < \widetilde{R} < +\infty$, the estimator $\widehat{\beta}_n(X;\widetilde{R})$ of parameter $\beta^\star$ is strongly consistent.*

Asymptotic normality of the estimator

Proposition

Let X be stationary (ergodic) Gibbs point process, under the assumptions **[FR]** *and* **[CH]**. *Then we have, for any fixed $0 < R \leq \widetilde{R} < +\infty$, as $n \to +\infty$ and as $|\Lambda_n| \to +\infty$,*

$$\sqrt{|\Lambda_n|}\left(\widehat{\beta}_n(X;\widetilde{R}) - \beta^\star\right) \xrightarrow{d} \mathcal{N}(0, \sigma^2(\beta^\star)),$$

where

- $\sigma^2(\beta^\star) = \dfrac{\sum_{j \in B(0,1)} \mathbf{E}\left[I_{\Delta_0(R)}(X, h; \beta^\star)\, I_{\Delta_j(R)}(X, h; \beta^\star)\right]}{R^d\left(1 - F(\widetilde{R})\right)^2},$
- $I_{\Delta_k(R)}(X, h, \beta^\star) = \sum_{u \in X_{\Delta_k(R)}} h(u, X \backslash u) - \int_{\Delta_k(R)} h(u, X)\lambda(u, X) du.$

By using the results of JF.Coeurjolly and E. Rubak (2012), we can calculate the value $\sigma^2(\beta^\star)$ differently as follows:

Proposition

Let X be stationary Gibbs point process. Under the assumptions **[FR]** *and* **[CH]**, *we have for any fixed $0 < R < \widetilde{R} < +\infty$,*

$$\sigma^2(\beta^\star) = \frac{\beta^\star(1 - F(\widetilde{R})) + {\beta^\star}^2 \int_{B(0,\widetilde{R})} (1 - F_{0,v}(\widetilde{R}))dv}{\left(1 - F(\widetilde{R})\right)^2},$$

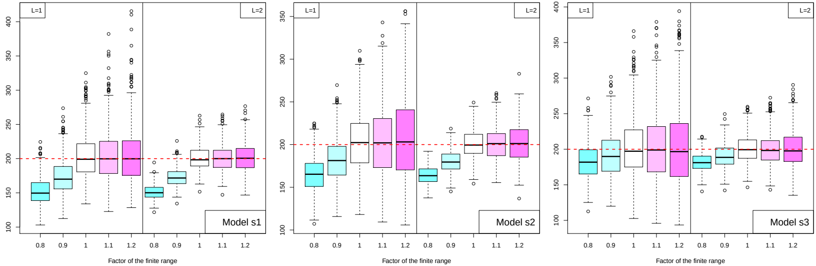
where

- $F(\widetilde{R}) = P_{\beta^\star}(X \cap B(0, \widetilde{R}) \neq \emptyset)$.
- $F_{0,v}(\widetilde{R}) = P_{\beta^\star}(X \cap B(0, \widetilde{R}) \neq \emptyset, X \cap B(v, \widetilde{R}) \neq \emptyset)$.

Simulation study

Strauss point process: $\lambda(u, x) = \beta\gamma^{n_{[0,R]}(u,x)}$

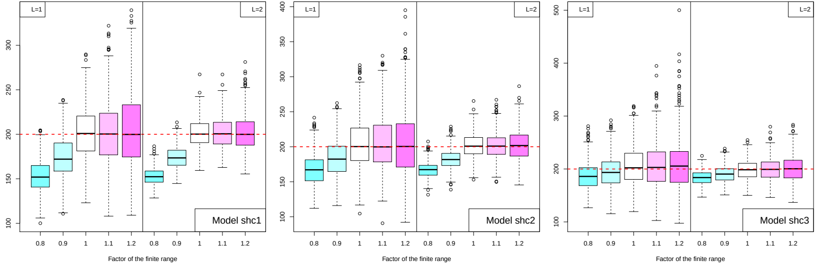
- s1: $\beta = 200$, $\gamma = 0.2$, $R = \varphi$.
- s2: $\beta = 200$, $\gamma = 0.5$, $R = \varphi$.
- s3: $\beta = 200$, $\gamma = 0.8$, $R = \varphi$.



Boxplots of the Poisson intensity parameter estimates for different parameters $\widetilde{R}$ from 0.8 to 1.2 times the finite range parameter φ, from 500 replications of the models s1,s2,s3 generated on the window $[0, L]^2 \oplus 1.2\varphi$ and estimated on the window $[0, L]^2$ for $L = 1, 2$.

Strauss point process with Hard-Core: $\lambda(u,x) = \beta\gamma^{n_{[0,R]}(u,x)}$

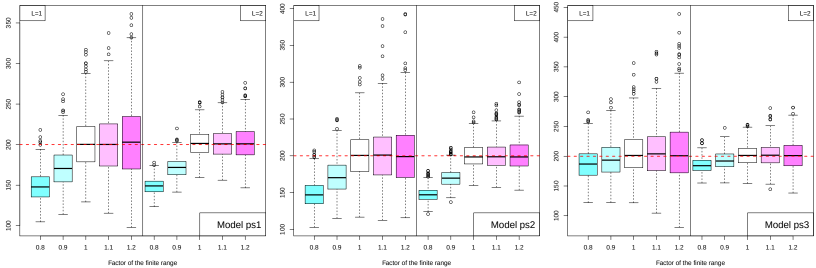
- shc1: $\beta = 200,\ \gamma = 0.2,\ \delta = \varphi/2,\ R = \varphi$.
- shc2: $\beta = 200,\ \gamma = 0.5,\ \delta = \varphi/2,\ R = \varphi$.
- shc3: $\beta = 200,\ \gamma = 0.8,\ \delta = \varphi/2,\ R = \varphi$.



Boxplots of the Poisson intensity parameter estimates for different parameters $\widetilde{R}$ from 0.8 to 1.2 times the finite range parameter φ, from 500 replications of the models shc1, shc2, shc3 generated on the window $[0, L]^2 \oplus 1.2\varphi$ and estimated on the window $[0, L]^2$ for $L = 1, 2$.

Piecewise Strauss point process: $\lambda(u,x) = \beta \prod_{j=1}^{p} \gamma_j^{n_{[R_{j-1},R_j]}(u,x)}$

- ps1: $\beta = 200$, $\gamma = (0.8, 0.5, 0.2)$, $R = (\varphi/3, 2/3\varphi, \varphi)$.
- ps2: $\beta = 200$, $\gamma = (0.2, 0.8, 0.2)$, $R = (\varphi/3, 2/3\varphi, \varphi)$.
- ps3: $\beta = 200$, $\gamma = (0.8, 0.5, 0.2)$, $R = (\varphi/3, 2/3\varphi, \varphi)$.



Boxplots of the Poisson intensity parameter estimates for different parameters $\widetilde{R}$ from 0.8 to 1.2 times the finite range parameter φ, from 500 replications of the models ps1,ps2,ps3 generated on the window $[0, L]^2 \oplus 1.2\varphi$ and estimated on the window $[0, L]^2$ for $L = 1, 2$.

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Remark

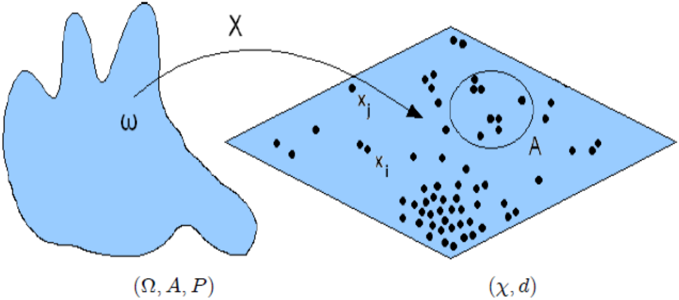

Figure 1

- We will not discuss how to consistently specify the Papangelou conditional intensity to ensure the existence of a Gibbs point process on $\mathbb{R}^d$, but rather we simply assume we are given a well-defined Gibbs point process.

- $\lambda(u,x) = \prod_{y \subseteq x \setminus u} \phi(y \cup u), x \in N_{lf}, u \in \mathbb{R}^d$

- La condition de portée: contrôle la répartition des points dans le voisinage des grandes arêtes de x .

Lemma

(ergodic)
We assume that X is ergodic point process. Then for any family of measurable functions F_Λ , indexed by the bounded sets Λ, from Ω (valued random variable) to $\mathbb{R}$ which are additive (i.e. $F_{\Lambda\cup\Lambda'} = F_\Lambda + F_{\Lambda'} - F_{\Lambda\cap\Lambda'}$), shift invariant (i.e. $F_\Lambda(X) = F_{\tau(\Lambda)}(\tau(X))$ for any translation τ) and integrable (i.e.$E\left[|F_{\Lambda_0}(X)|\right] < +\infty$), we have

$$\lim_{n\to+\infty} |\Lambda_n|^{-1} F_{\Lambda_n}(X) = E\left[F_{\Lambda_0}(X)\right], \quad a.s.$$

where $\Lambda_n = [-n, n]^d$ (other regular domains $(\Lambda_n)_{n\geq 1}$ converging towards $\mathbb{R}^d$ could be also considered).

We assume that P_{β^*} is Gibbs measure of the stationary Gibbs point process X, where $\beta^* \in \mathring{\Theta}$ is the true parameter to be estimated and for any β denote a current point in Θ, there exists a stationary Gibbs measure P_β. If there is more than one stationary Gibbs measure, then some non ergodic Gibbs measures automatically exist because, in the convex set of all Gibbs measures, only the extremal measures are ergodic. But any stationary Gibbs measure can be represented as a mixture of ergodic measures. Due to this decomposition, we can assume that P_{β^*} is ergodic to prove the consistency of our estimator.

Theorem

Let $X_{n,i}$, $n \in \mathbb{N}$, $i \in \mathbb{Z}^d$, be a triangular array field in a measurable space S. For $n \in \mathbb{N}$, let $\mathcal{K}_n \subset \mathbb{Z}^d$ and for $k \in \mathcal{K}_n$, assume

$$Z_{n,k} = f_{n,k}\left(X_{n,k+i},\ i \in \mathcal{I}_0\right), \tag{2}$$

where $\mathcal{I}_0 = \{i \in \mathbb{Z}^d,\ |i| \leq 1\}$ and $f_{n,k} : S^{\mathcal{I}_0} \to \mathbb{R}^p$. Let $S_n = \sum_{k \in \mathcal{K}_n} Z_{n,k}$. If

(i) $c_3 := \sup_{n\in\mathbb{N}} \sup_{k\in\mathcal{K}_n} |Z_{n,k}|^3 < +\infty$, Processus ponctuel d'interaction paires:

$$\lambda(u,x) = \beta \gamma^{\sum_{v \in x} g(||v-u||)}.$$