Second order finite elements for the SPDE approach

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RESSTE SPDE/INLA workshop

Stochastic partial differential equation

Whittle [1963]

If $Z(x)$ is a stationary random function in d-space with Matérn covariance with regularity ν, scale parameter κ and variance σ^2 then $Z(x)$ is solution of the SPDE

$$(\kappa^2 - \Delta)^{\alpha/2} Z(x) = \mathcal{W}(x), \; x \in \Omega$$

with

- $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian
- $\mathcal{W}(x)$ is a standardized white noise process
- $\alpha = \nu + d/2$
- $\sigma^2 = \frac{\Gamma(\nu)}{\Gamma(\nu + d/2)(4\pi)^{d/2}\kappa^{2\nu}}$

Practically defined with Neumann type limit conditions on $\partial\Omega$

Weak formulation

case α=2, $d = 2$

$$\left(\kappa^2 - \Delta\right) Z(x) = \mathcal{W}(x)$$

Let

- the inner product $\langle f, g\rangle = \int_\Omega f(x)g(x)\mathrm{d}x$
- ϕ an appropriate **test** function (typically $\mathcal{C}_c^\infty$)

By integrating over Ω

$$\kappa^2\langle Z, \phi\rangle + \langle \nabla Z, \nabla\phi\rangle = \langle \mathcal{W}, \phi\rangle$$

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Discretization

We approximate the solution space by span$\{\psi_i(x), i = 1 \dots N\}$
Therefore $Z(x) = \sum_{i=1}^N z_i \psi_i(x)$ and we get the equation

$$\sum_{i=1}^{N} z_i \left(\kappa^2 \langle \psi_i, \psi_j \rangle + \langle \nabla \psi_i, \nabla \psi_j \rangle\right) = \langle \mathcal{W}, \psi_j \rangle, j = 1 \dots N$$

Denoting

- $C_{i,j} = \langle \psi_i, \psi_j \rangle$ the mass matrix
- $G_{i,j} = \langle \nabla \psi_i, \nabla \psi_j \rangle$ the stiffness matrix
- $K = \kappa^2 C + G$

$\mathrm{Z} = (z_1, \dots, z_N)$ is solution of the linear system

$$KZ = (W\Psi),$$

where $(W\Psi)_j = \langle \mathcal{W}, \psi_j \rangle, j = 1 \dots N$

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Gaussian Markov random field

We clearly have

$$\mathbb{E}(Z) = 0$$

and computing the **covariance** we get

$$K\Sigma K = C,$$

where Σ is the covariance matrix of Z

The **precision** matrix of Z is therefore

$$Q = \Sigma^{-1} = KC^{-1}K$$

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Gaussian Markov random field

Lindgren et al. [2011]

More generally, if the SPDE writes

$$P^{1/2}(-\Delta)Z = \mathcal{W}(x),\ x \in \Omega$$

Then the precision matrix is

$$Q = C^{1/2}P(M)C^{1/2}$$

where $M = C^{-1/2}GC^{-1/2}$

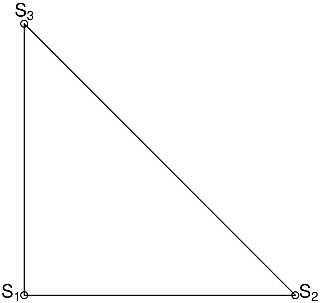
Hence **mass lumping** is essential

Lagrange P1 elements

The random function Z is approximated by

$$Z(x) = \sum_{i=1}^{N} z_i \psi_i(x)$$

where ψ_i is equal to 1 at vertice i and decreases linearly to 0 at the neighbouring vertices (Lagrange P_1 element)



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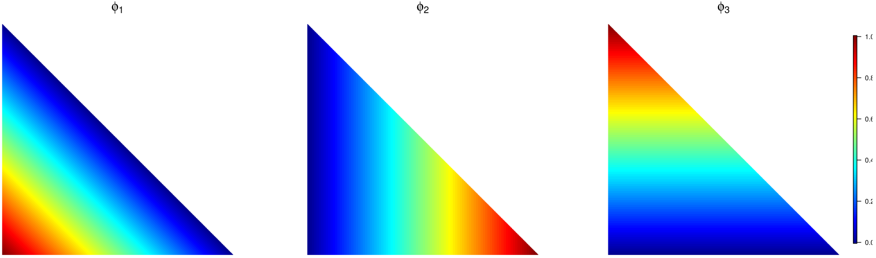

Mass lumping

Cohen et al. [2001]

Let T a triangle, to get a diagonal mass matrix we replace

$$C_{ij} = \langle \psi_i, \psi_j \rangle = \int_T \psi_i(x)\psi_j(x)dx$$

by a quadrature formula

$$\widetilde{C}_{ij} = \sum_i w_i \psi_i(\xi_i)\psi_j(\xi_i)$$

where ξ_i and w_i are respectively the nodes and the weights

If the quadrature points coincide with the nodes of the finite element space, then $\widetilde{C}$ is diagonal

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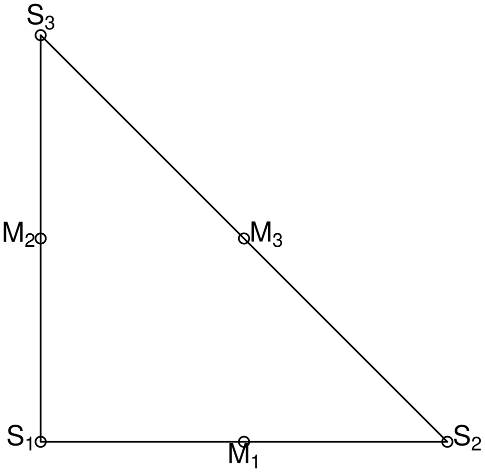
Lagrange P_2 finite element



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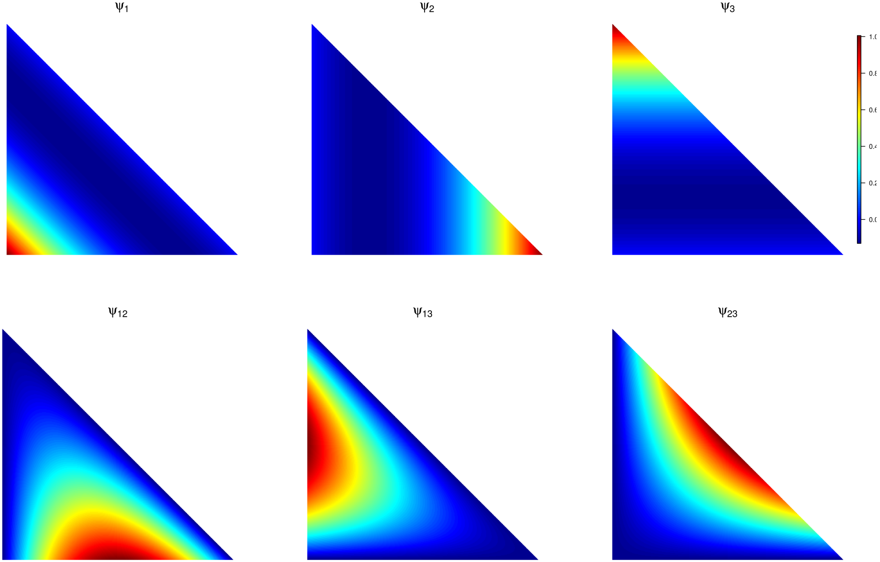

Does mass lumping occur?

Cohen et al. [2001]

We need a sufficiently accurate quadrature formula

- In each triangle T, it must be exact for P_2
- It must be symmetric
- The set of quadrature points should be P_2 unisolvent

The solution is

$$w_s = 0 \text{ and } w_M = 1/3$$

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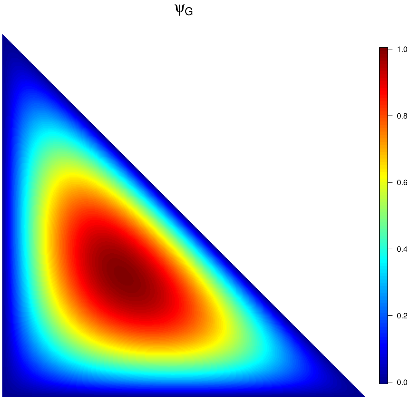
New finite element space

Cohen et al. [2001]

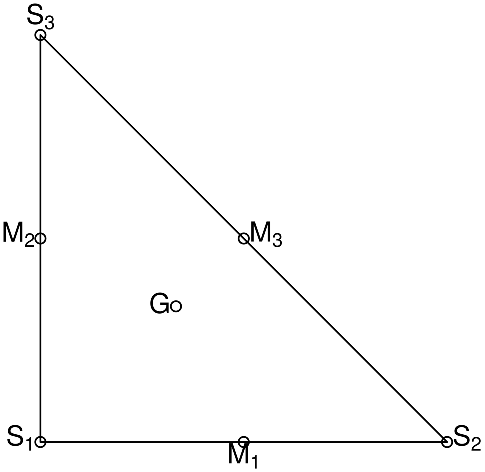
The solution is to work with a slightly larger finite element space

$$\widetilde{P}_2 = P_2 \oplus [\psi_G]$$

where ψ_G is the "bubble" function



Lagrange $\widetilde{P}_2$ finite element



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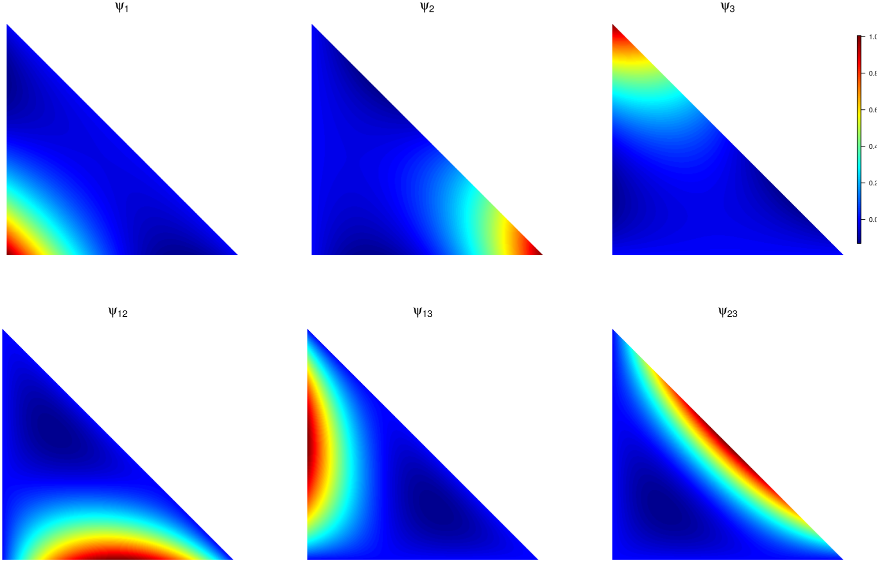


Quadrature formula

Cohen et al. [2001]

To get the same accuracy with $\widetilde{P}_2$ as in standard P_2 elements, the quadrature formula should be exact in P_3

This leads to the weights

$$w_s = 1/20, w_M = 2/15 \text{ and } w_G = 9/20$$

which is apparently well known as Simpson's rule

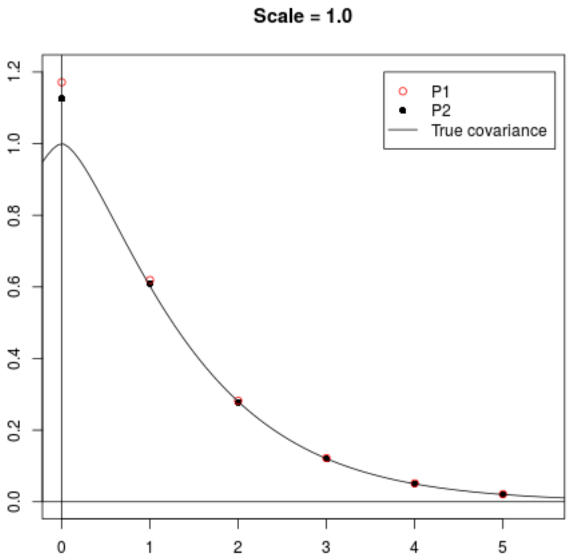
Illustration

We compare how well the covariance is reproduced using P_1 and P_2 FEMs

We increase progressively the scale parameter to mimic the asymptotic behaviour

The regularity parameter is set to 1

Illustration



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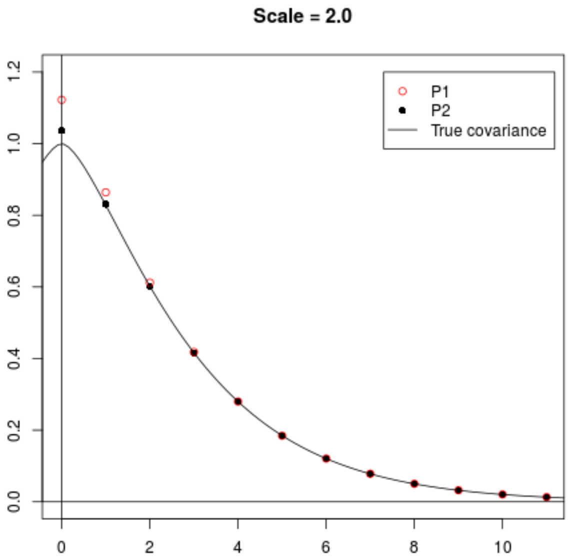

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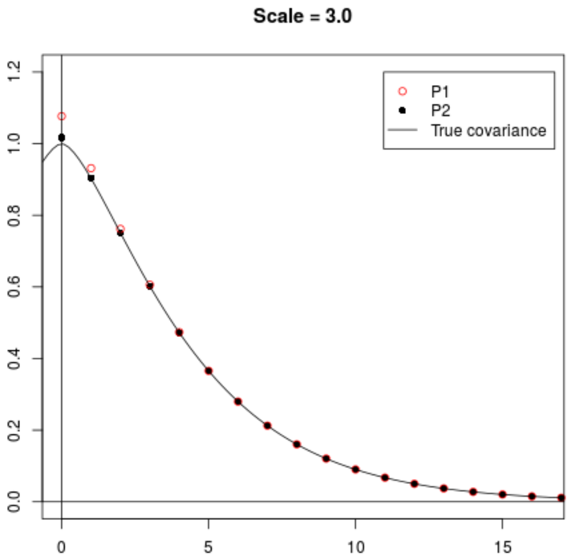

Conclusion

- Mass lumping is available for higher order finite elements at the cost of expanded finite element space
- The practical interest of using second order finite elements remains to investigate
- 3D?

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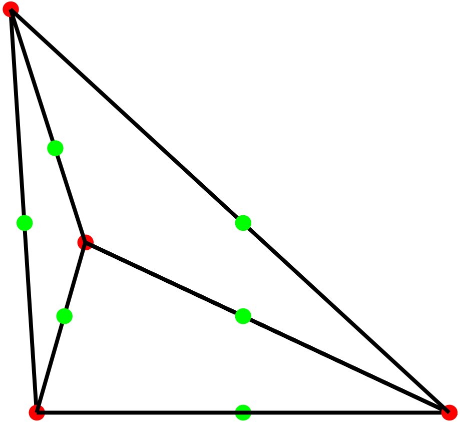
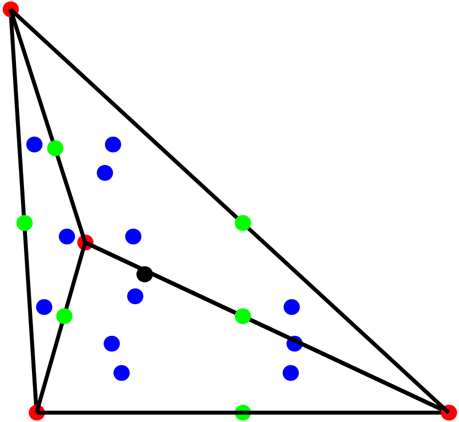
3D



10 degrees of freedom

3D



23 degrees of freedom!

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