Space-Time Covariance functions for Planet Earth

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State of the Art: Space-time Gaussian Fields

- $\{Z(\mathbf{s}, t), \mathbf{s} \in D \subset \mathbb{R}^d, \quad t \in \mathbb{R}\}$ Gaussian random fields (GRF).
- Covariance function

$$Cov(Z(\mathbf{s}_1, t_1), Z(\mathbf{s}_2, t_2))$$

- Weak Stationarity

$$E(Z(\mathbf{s})) = 0$$

$$\mathrm{Cov}(Z(\mathbf{s}_1, t_1), Z(\mathbf{s}_2, t_2)) = C(\mathbf{s}_1 - \mathbf{s}_2, t_1 - t_2) =: C(\mathbf{h}, u)$$

($\mathbf{h} = \mathbf{s}_1 - \mathbf{s}_2$, spatial lag, and $u := t_1 - t_2$ temporal lag).

State of the Art: Space-time Gaussian Fields

- $C(\mathbf{h}, u) = \int \mathrm{e}^{i(\boldsymbol{\omega},\mathbf{h})+i(\tau,u)}\mu(\mathrm{d}(\boldsymbol{\omega},\tau))$
- Separability $C(\mathbf{h}, u) = C_1(\mathbf{h})C_2(u)$
- Full Symmetry $C(\mathbf{h}, u) = C(-\mathbf{h}, u) = C(-\mathbf{h}, -u) = C(\mathbf{h}, -u)$

Stationarity and Isotropy

- Spatial Isotropy (Daley and Porcu, PAMS, 2013)

$$C(\mathbf{h}, 0) = K(\|\mathbf{h}\|, 0)$$

$K : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ such that

$$K(r, 0) = \int_{[0,\infty)} \Omega_d(r\xi) F(\mathrm{d}\xi), \qquad r \geq 0,$$

$$\Omega_d(t) = t^{-(d-2)/2} J_{(d-2)/2}(t), \qquad t \geq 0.$$

- We suppose that K comes from some parametric family $\mathcal{C}_\theta$

A popular class

- The Gneiting class (Gneiting, 2002; Zastavnyi and Porcu, 2011)

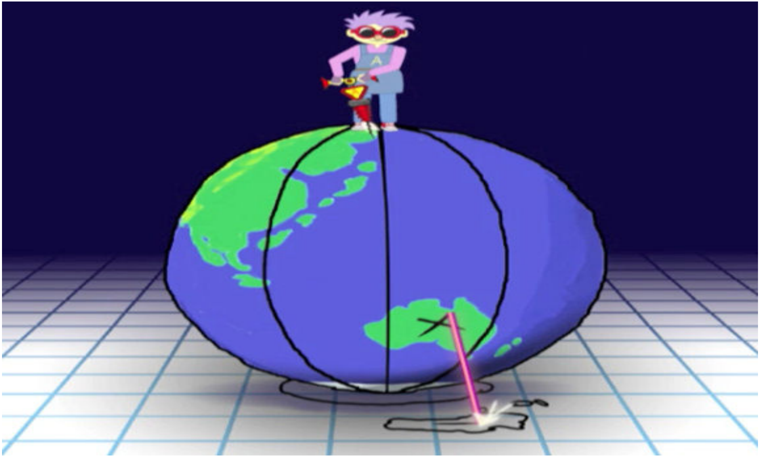
$$C(\mathbf{h}, u) = \frac{\sigma^2}{\psi(u^2)^{-d/2}} \varphi \left(\frac{\|\boldsymbol{h}\|^2}{\psi(u^2)} \right)$$

- Example: $\varphi(t) = \mathrm{e}^{-t}$, $\psi(t) = (1 + t)^{-1}$

$$C(\mathbf{h}, u) = \frac{\sigma^2}{\left(1 + \left(\frac{u}{c_t}\right)^2\right)^{-d/2}} \exp \left(\frac{\|\boldsymbol{h}\|^2}{c_s \left(1 + \left(\frac{u}{c_t}\right)^2\right)} \right)$$

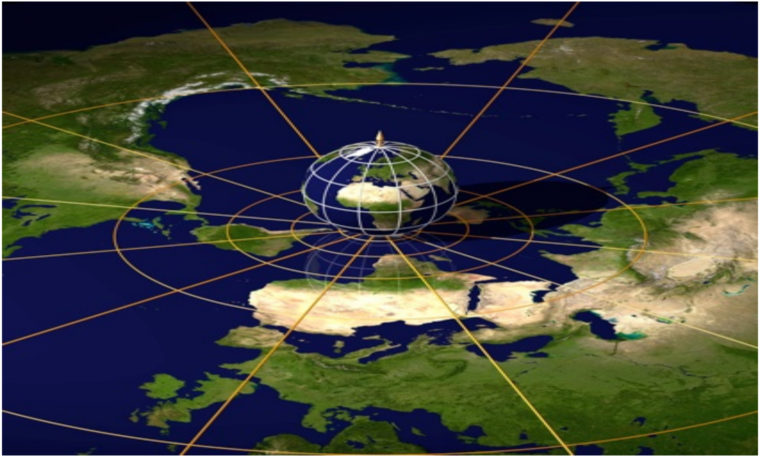
Spheres

Spatial Stats and the Euclidean Paradigm



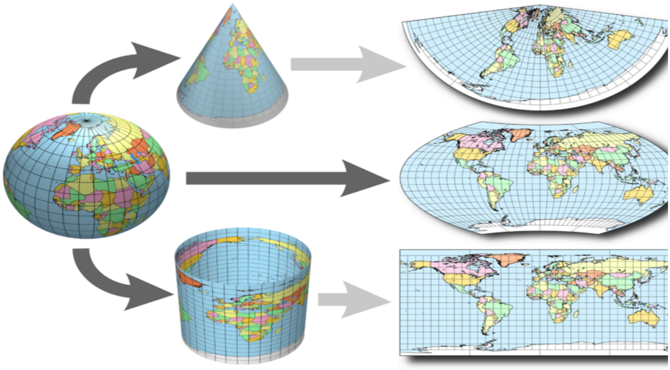
Spheres

Spatial Stats and the Euclidean Paradigm



Spheres

Spatial Stats and the Euclidean Paradigm

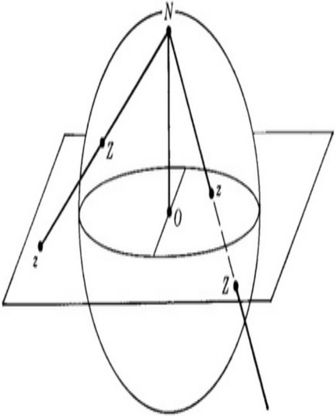


Spheres

Chordal and Great Circle

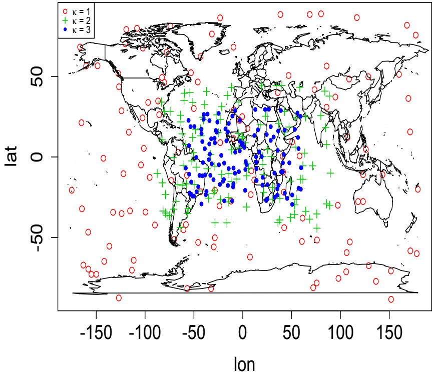
Chordal distance: criticism on

- Negative Correlations
- Counter to Spherical geometry



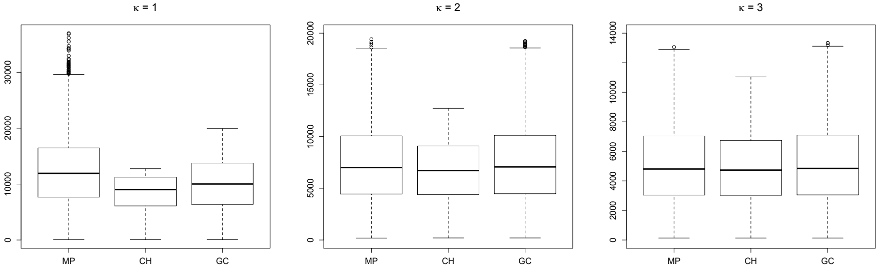
Spheres

A toy Example



Spheres

A toy example (2)



Word of Caution

- Suppose K is a isotropic covariance function. Can we replace the Euclidean with the geodesic distance?
- The answer is NOT.
- Example:

$$C(\mathbf{h}) = (\alpha\|\mathbf{h}\|)^{\nu}\mathcal{K}_{\nu}(\alpha\|\mathbf{h}\|), \qquad \mathbf{h} \in \mathbb{R}^{d}$$

Spheres

Space-Time Challenges

Frequently, the temporal development of a process observed on a sphere is also of interest, so that the process needs to be modeled on the sphere cross time. Nevertheless, the literature on the corresponding correlation structures is sparse [...]
Tilmann Gneiting, *Problem 16* of Online supplement to *Bernoullli* (2013).

Outline

1. motivations
2. Schoenberg coefficients and functions
3. The class $\Psi_{d,T}$
4. Construction Principles
5. Assessing discrepancies between the great circle distance and other metrics
6. Analysis of TOMS data

Based on

1. **Porcu, E.**, Bevilacqua, M. & Genton, M.G. (2015). *Journal of the American Statistical Association*. To appear.
2. Berg, C. & **Porcu, E.** *Constructive Approximation*. Accepted.

Spheres

The sphere

- d-dimensional unit sphere of $\mathbb{R}^{d+1}$, given as

$$\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} \mid ||x|| = 1\}, \; d \geq 1.$$

- Great Circle distance: $\theta : \mathbb{S}^d \times \mathbb{S}^d \to [0, \pi]$,

$$\theta(\xi, \eta) = \arccos(\xi \cdot \eta),$$

- Chordal Distance: $\mathrm{ch}(\xi, \eta) = 2 \sin\left(\frac{\theta}{2}\right)$.

Motivations

Gaussian fields on $\mathbb{S}^d \times \mathbb{R}$

- Stationary Gaussian fields $\{Z(\xi, t), \ (\xi, t) \in \mathbb{S}^d \times \mathbb{R}\}$,
- Covariance functions $C : \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}$, so that

$$\boxed{\mathrm{cov}\left(Z(\xi, t), Z(\eta, t')\right) = C(\theta(\xi, \eta), t - t')}, \qquad (\xi, t), (\eta, t') \in \mathbb{S}^d \times \mathbb{R}.$$

- Class $\Psi_{d,T}$ of continuous functions $f : [-1, 1] \times \mathbb{R} \to \mathbb{C}$ such that C can be written as

$$C(\theta(\xi, \eta), t - t') = \boxed{f(\cos\theta(\xi, \eta), t - t')}, \quad \xi, \eta \in \mathbb{S}^d, \ t, t' \in \mathbb{R}. \tag{1}$$

Spheres and Schoenberg Class

An Intermezzo: The Class Ψ_d

- We also consider

$$\Psi_{\infty,T} := \bigcap_{d=1}^{\infty} \Psi_{d,T},$$

$$\mathbb{S}^{\infty} = \{(x_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} | \sum_{k=1}^{\infty} x_k^2 = 1\},$$

  which is the unit sphere in the Hilbert sequence space ℓ_2 of square summable real sequences.

- Inclusion Relation

$$\Psi_{1,T} \supset \Psi_{2,T} \supset \ldots \supset \Psi_{\infty,T}.$$

Spheres and Schoenberg Class

An Intermezzo: The Class Ψ_d

- The class $\Psi_{d,T}$ is parenthetical to the class Ψ_d of continuous functions $\psi : [0, \pi] \to \mathbb{R}$ such that $\psi(\theta)$ is a covariance function on $\mathbb{S}^d \times \mathbb{S}^d$.

Spheres

Schoenberg's Class: an Intermezzo

Iso Schoenberg

21-04-1903 in Galati, 21-04-1990

- Son of a medical doctor
- 1922: M.A. at Jessy University
- 1922: Goettingen (Schur)
- 1925: Edmund Landau

1930: Harvard and Princeton

- *the isometric imbedding of metric spaces into Hilbert space and positive definite functions.*
- 1950: Pólya
- 1966: University of Pennsylvania

He was...[by Richard Askey]

a man of broad culture, fluent in several languages, addicted to art, music and world literature, sensitive, gracious and giving in all ways. The working desk at his home where he engages in research is actually a draftsman's bench complete with T-square, etc. and a tall stool. Mobiles, artistic works, models of ruled surfaces, icosahedrons and other objects are strewn throughout the room. English, French and German novels, numerous paintings and artefacts are scattered on all the nearby easy chairs.

Spheres: how to build the Class Ψ_d

The Class Ψ_d: How to build it

- Gegenbauer polynomials

$$(1-2xr+r^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x) r^n, \quad |r|<1, x \in \mathbb{C}.$$

- For $\lambda > 0$,

$$\int_{-1}^{1} (1-x^2)^{\lambda-1/2} C_n^{(\lambda)}(x) C_m^{(\lambda)}(x)\,\mathrm{d}x = \frac{\pi \Gamma(n+2\lambda) 2^{1-2\lambda}}{\Gamma^2(\lambda)(n+\lambda) n!} \delta_{m,n}.$$

Spheres: how to build the Class Ψ_d

The Class Ψ_d

- Important!

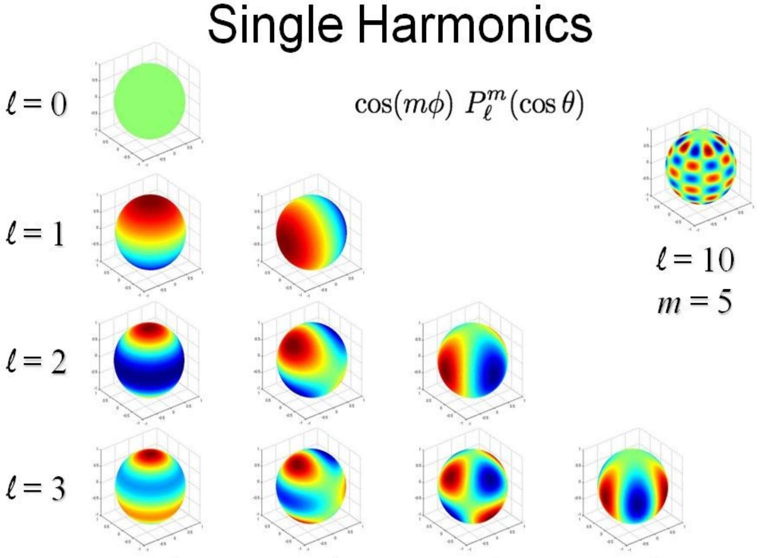
$$|C_n^{(\lambda)}(x)| \le C_n^{(\lambda)}(1), \quad x \in [-1,1].$$

- $\lambda = (d-1)/2$ and its connection with spherical harmonics.
- $n \in \mathbb{N}$. Vector Space $\boxed{\mathcal{H}_n(d) \subset \mathcal{C}(\mathbb{S}^d)}$, dimension

$$N_n(d) := \dim \mathcal{H}_n(d) = \frac{(d)_{n-1}}{n!}(2n+d-1), \; n \ge 1, \quad N_0(d) = 1,$$

Spheres: how to build the Class Ψ_d

The Class $\Psi_{d,T}$



Spheres: how to build the Class Ψ_d

The Class Ψ_d

- Orthogonality relation:

$$\int_{-1}^{1}(1-x^2)^{d/2-1}c_n(d,x)c_m(d,x)\,\mathrm{d}x = \frac{||\omega_d||}{||\omega_{d-1}||N_n(d)}\delta_{m,n}.$$

- $\boxed{Theorem(Schoenberg, 1942)}$ A continuous function $f:[-1,1]\to\mathbb{R}$ belongs to the class Ψ_d, $d=1,2,\ldots$, if and only if

$$f(\cos\theta)=\sum_{n=0}^{\infty}b_{n,d}c_n(d,\cos\theta), \quad b_{n,d}\geq 0, \theta\in[0,\pi],$$

for a summable sequence $(b_{n,d})_{n=0}^{\infty}$ given as

$$b_{n,d}=\frac{||\omega_{d-1}||N_n(d)}{||\omega_d||}\int_{-1}^{1}f(x)c_n(d,x)(1-x^2)^{d/2-1}\,\mathrm{d}x.$$

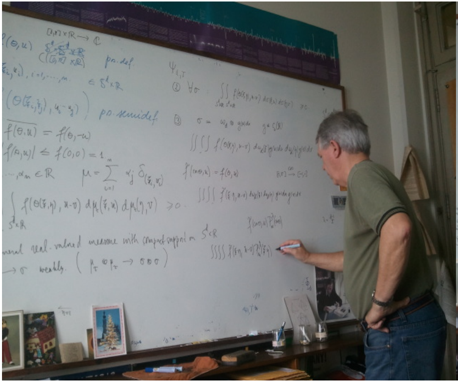
Spheres: how to build the Class Ψ_d

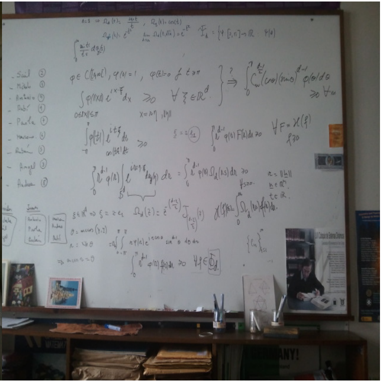
The Class $\Psi_{d,T}$

- If $f(0) = 1$, $(b_{n,d})$ is a probability sequence
- Daley and Porcu (2013) call $b_{n,d}$ d-Schoenberg coefficients and the sequence $(b_{n,d})$ a d-Schoenberg sequence
- In $d = 1$,

$$f(\cos\theta) = \sum_{n=0}^{\infty} b_{n,1} \cos(n\theta), \quad b_{n,1} \geq 0, \theta \in [0, \pi],$$

Characterization of $\Psi_{d,T}$





Results

The Class $\Psi_{d,T}$: Characterization Theorems

Theorem 1 Let $d \in \mathbb{N}$ and let $f : [-1,1] \times \mathbb{R} \to \mathbb{C}$ be a continuous function. Then f belongs to $\Psi_{d,T}$ if and only if there exists a sequence $\varphi_{n,d} : \mathbb{R} \to \mathbb{R}$ of p.d. functions with $\sum \varphi_{n,d}(0) < \infty$ such that

$$f(\cos\theta, t) = \sum_{n=0}^{\infty} \varphi_{n,d}(t) c_n(d, \cos\theta),$$

and the above expansion is uniformly convergent for $(\theta, t) \in [0, \pi] \times \mathbb{R}$. We have

$$\varphi_{n,d}(t) = \frac{N_n(d) ||\omega_{d-1}||}{||\omega_d||} \int_{-1}^{1} f(x,t) c_n(d,x) (1-x^2)^{d/2-1} \, \mathrm{d}x.$$

Construction principles

with Moreno Bevilacqua and Marc Genton

Results

The Class $\Psi_{d,T}$

A natural construction:

$$Z(\eta, t) = \sum_{k=0}^{\infty} \sum_{\nu \in \Upsilon_{k,d}} \xi_{k,\nu}(t) Y_{k,\nu,d}(\eta), \qquad \eta \in \mathbb{S}^d, t \in \mathbb{R},$$

- $Y_{k,\nu,d} : \mathbb{S}^d \to \mathbb{C}$: normalized hyperspherical harmonics;
- $\xi_{k,\nu}(t)$: Gaussian processes, with zero mean and $\mathbb{E}\xi_{k,\nu}(t)\xi_{k',\nu'}(t') = \delta_{k,k'}\delta_{\nu,\nu'} g_k(t - t')$, $t, t' \in \mathbb{R}$.

Construction Principles

An easy construction principle

Let $\{g_k(\cdot)\}_{k=0}^{\infty}$ be an absolutely convergent sequence of continuous and positive definite functions on the real line, such that $g_k(0) = b_k$ for all $k = 0, 1, \ldots$, with $\{b_k\}_{k=0}^{\infty}$ being a probability mass sequence. Then,

$$C(\theta, u) = \sum_{k=0}^{\infty} g_k(u) \left(\cos \theta\right)^k, \qquad (\theta, u) \in [0, \pi] \times \mathbb{R},$$

is a representation for members of the class $\Psi_{\infty,T}$.

Results

The Class $\Psi_{d,T}$

Family	Analytic expression	Parameters range
Negative Binomial	$C(\theta, u) = \left\{ \frac{1-\varepsilon}{1-\varepsilon g(u) \cos\theta} \right\}^{\tau}$	$\varepsilon \in (0,1), \ \tau > 0$
Multiquadric	$C(\theta, u) = \frac{(1-\varepsilon)^{2\tau}}{\{1+\varepsilon^2 - 2\varepsilon g(u) \cos\theta\}^{\tau}}$	$\varepsilon \in (0,1), \ \tau > 0$
Sine Series	$C(\theta, u) = e^{g(u)\cos\theta - 1} \{1 + g(u)\cos\theta\}/2$	
Sine Power	$C(\theta, u) = 1 - 2^{-\alpha}\{1 - g(u)\cos\theta\}^{\alpha/2}$	$\alpha \in (0, 2]$
Adapted Multiquadric	$C(\theta, u) = \left[\frac{\{1+g^2(u)\}(1-\varepsilon)}{1+g^2(u) - 2\varepsilon g(u)\cos\theta} \right]^{\tau}$	$\varepsilon \in (0,1), \tau > 0$ $2g(\cdot)/\{1 + g^2(\cdot)\}$ corr. function on $\mathbb{R}$
Poisson	$C(\theta, u) = \exp[\lambda\{\cos\theta g(u) - 1\}]$	$\lambda > 0$

Results

Spatial Adapting from the Gneiting class

- Gaussian process Z on $\mathbb{R}^d \times \mathbb{R}$, points $(\boldsymbol{x}, t)$ and $(\boldsymbol{y}, t')$ such that $\|\boldsymbol{y} - \boldsymbol{x}\| = \boldsymbol{h}$ (with $\|\cdot\|$ denoting the Euclidean distance) and $t - t' = u$,

$$C(\boldsymbol{h}, u) := \frac{\sigma^2}{\psi(\|\boldsymbol{h}\|^2)^{d/2}} \varphi \left\{ \frac{u^2}{\psi(\|\boldsymbol{h}\|^2)} \right\}, \qquad (\boldsymbol{h}, u) \in \mathbb{R}^d \times \mathbb{R},$$

- φ is completely monotone on the positive real line such that $\varphi(0) = 1$, ψ is a positive-valued Bernstein function, and σ^2 is a variance parameter.

Results

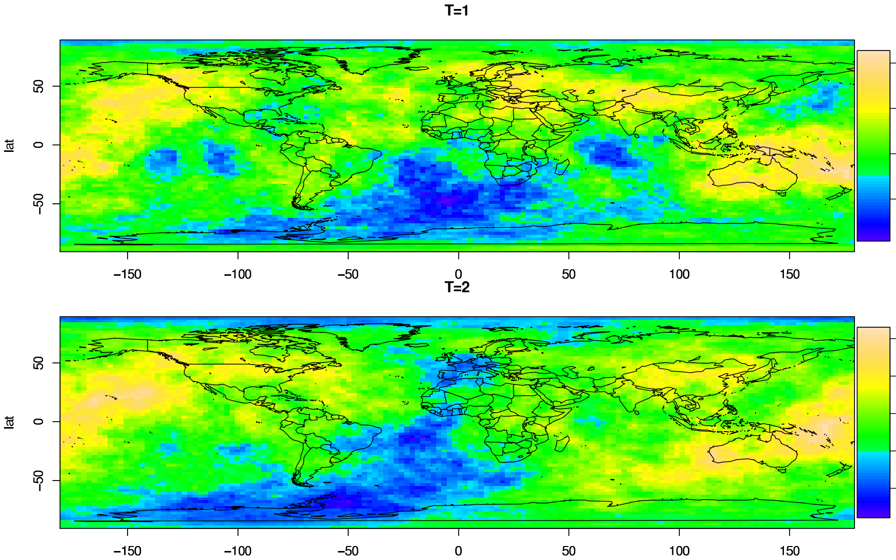
Spatial Adapting from the Gneiting class

Let $\theta : \mathbb{S}^d \times \mathbb{S}^d \to [0, \pi]$ be the great circle distance. Let $\varphi : [0, \infty) \to \mathbb{R}_+$ be a completely monotone function on the positive real line, with $\varphi(0) = 1$, and let ψ be a positive-valued Bernstein function. Denote by $\psi_{[0,\pi]}$ the restriction of ψ to the interval $[0, \pi]$. Then, the function

$$C(\theta, u) := \frac{\sigma^2}{\psi_{[0,\pi]}(\theta)^{1/2}} \varphi \left\{ \frac{u^2}{\psi_{[0,\pi]}(\theta)} \right\}, \qquad (\theta, u) \in [0, \pi] \times \mathbb{R}, \qquad (2)$$

belongs to the class $\Psi_{\infty,T}$.

Gneiting class on the sphere



Results

Relevant Comments

- Mean square differentiability for processes on spheres. Attempts in Jeong and Jun (2015).
- Exception being the Sine Power model.
- Adapted construction has the same problem: Matérn is valid only for $\nu \in (0, 1/2]$.
- Adapted construction allows for rescaling the spatial component, direct construction not
- Direct construction allow for any type of temporal margin, provided g is a temporal correlation function.

Results

Examples from The Adapted Gneiting Class

- $$C(\theta, u) = \frac{\sigma^2}{\left\{1 + \left(\frac{R\theta}{c_S}\right)^{\alpha}\right\}^{\delta+\beta/2}} \exp\left[-\frac{\left(\frac{|u|}{c_T}\right)^{2\gamma}}{\left\{1 + \left(\frac{R\theta}{c_S}\right)^{\alpha}\right\}^{\beta\gamma}}\right],$$

- $$C(\theta, u) = \frac{\sigma^2}{\left\{1 + \left(\frac{R\theta}{c_S}\right)^{\alpha}\right\}^{\delta+\beta/2}} \left[1 + \frac{\left(\frac{|u|}{c_T}\right)^{2\tau}}{\left\{1 + \left(\frac{R\theta}{c_S}\right)^{\alpha}\right\}^{\tau\beta}}\right]^{-\lambda},$$

Results

Examples from The Adapted Gneiting Class

- Take the negative binomial family and $u \mapsto g(u;\alpha) := (1+|u|^{\alpha})^{-1}$, $\alpha \in (0,2]$,

$$C(\theta,u) = \sigma^2 \left[\frac{1-\varepsilon}{1-\varepsilon\left\{1+\left(\frac{|u|}{c_T}\right)^{\alpha}\right\}^{-1}\cos\theta} \right]^{\tau}, \qquad (\theta,u) \in [0,\pi]\times\mathbb{R},$$

- From the multiquadric,

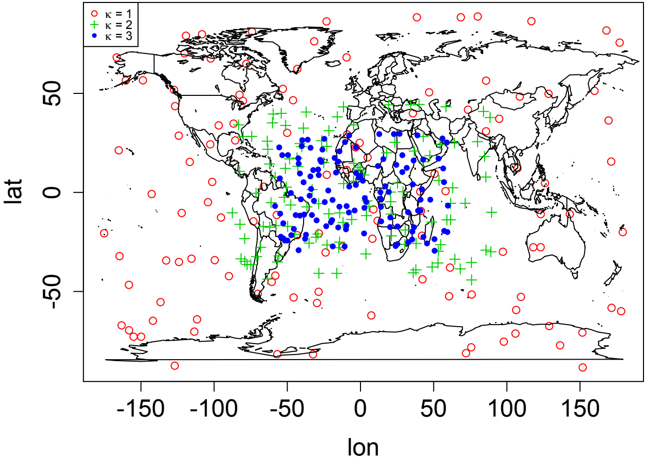
$$C(\theta,u) = \frac{\sigma^2(1-\varepsilon)^{2\tau}}{\left[1+\varepsilon^2-2\varepsilon\left\{1+\left(\frac{|u|}{c_T}\right)^{\alpha}\right\}^{-1}\cos\theta\right]^{\tau}}, \qquad (\theta,u) \in [0,\pi]\times\mathbb{R}, \tag{3}$$

with the same restriction on the parameters as in the previous model.

- $$C(\theta,u) = \sigma^2 \upsilon \left[1+\frac{\cos\theta}{c_S\left\{1+\left(\frac{|u|}{c_T}\right)^{\alpha}\right\}^{-1}}\right] \exp\left[\frac{\cos\theta}{c_S\left\{1+\left(\frac{|u|}{c_T}\right)^{\alpha}\right\}^{-1}}\right], \qquad (\theta,u) \in [0,\pi]\times\mathbb{R},$$

Simulation Studies

Scenarios



Simulation Studies

Scenarios

- Three scenarios;
- Estimate $\boldsymbol{\lambda}$ using ML;
- Under $C(\theta, u; \boldsymbol{\lambda}) = \frac{\sigma^2}{\left\{1+\left(\frac{R\theta}{c_S}\right)\right\}} \exp\left[-\frac{|u|}{c_T\left\{1+\left(\frac{R\theta}{c_S}\right)\right\}^{1/4}}\right]$,
- Using either the GC, CH or MP distances. Notation $\widehat{\boldsymbol{\lambda}}_{\mathcal{X}}^{(k)}$, with $\mathcal{X} =$ GC, CH or MP
- $k = 1, \ldots, 1000$ Simulations.

Simulation Studies

Scenarios

Discrepancy between the ML estimates using either GC, CH and MP distances. Given $\widehat{\boldsymbol{\lambda}}_{\mathcal{X}}^{(k)}$, we call $M(\cdot)$ the measure

$$M^{\mathcal{X}}({}_i\widehat{\lambda}) = \sqrt{\frac{\sum_{k=1}^{1000}({}_i\widehat{\lambda}_{\mathrm{GC}}^{(k)} - {}_i\widehat{\lambda}_{\mathcal{X}}^{(k)})^2}{1000}}, \qquad i = 1, 2, 3, \qquad \mathcal{X} = \mathrm{CH}, \mathrm{MP}. \tag{4}$$

We also define another measure $A(\cdot)$ by

$$A^{\mathcal{X}}({}_i\widehat{\lambda}) = \sqrt{\frac{\sum_{k=1}^{1000}({}_i\widehat{\lambda}_{\mathcal{X}}^{(k)} - {}_i\lambda)^2}{1000}}, \qquad i = 1, 2, 3, \qquad \mathcal{X} = \mathrm{GC}, \mathrm{CH}, \mathrm{MP}, \tag{5}$$

where ${}_i\lambda$ denotes the nominal value chosen under one of the proposed scenarios.

Simulation Studies

Scenarios

		$\kappa = 1$		$\kappa = 2$		$\kappa = 3$
	(I)	(II)	(I)	(II)	(I)	(II)
$M^{CH}(\hat{c}_S)$	6.25	24.07	0.97	4.06	0.42	2.19
$M^{MP}(\hat{c}_S)$	93.03	201.99	9.86	17.45	3.17	5.91
$M^{CH}(\hat{c}_T)$	0.013	0.014	0.004	0.008	0.004	0.007
$M^{MP}(\hat{c}_T)$	0.058	0.080	0.015	0.022	0.007	0.012
$M^{CH}(\hat{\sigma}^2)$	0.004	0.011	0.001	0.003	0.001	0.002
$M^{MP}(\hat{\sigma}^2)$	0.026	0.045	0.005	0.008	0.002	0.004
$A^{GC}(\hat{c}_S)$	116.47	206.77	67.74	111.52	45.67	74.90
$A^{CH}(\hat{c}_S)$	117.73	211.78	68.00	112.34	45.73	74.98
$A^{MP}(\hat{c}_S)$	133.73	261.39	68.59	112.96	45.89	74.16
$A^{GC}(\hat{c}_T)$	0.212	0.211	0.212	0.212	0.212	0.211
$A^{CH}(\hat{c}_T)$	0.212	0.212	0.212	0.212	0.212	0.212
$A^{MP}(\hat{c}_T)$	0.220	0.226	0.213	0.213	0.213	0.212
$A^{GC}(\hat{\sigma}^2)$	0.088	0.101	0.084	0.094	0.083	0.093
$A^{CH}(\hat{\sigma}^2)$	0.088	0.104	0.084	0.094	0.083	0.093
$A^{MP}(\hat{\sigma}^2)$	0.089	0.105	0.084	0.095	0.083	0.093

TOMS DATA

Scenario

- Level-3 Total Ozone Mapping Spectrometer (TOMS): daily total column ozone levels.
- Spatially irregular grid (1^o latitude by 1.25^o longitude away from the poles)
- Original data: Latitude interval $[-89.5, 89.5]$ and longitudes $[-180, 180]$
- Jun and Stein (2008): spatial dataset
- Here: 15 obs. in time, for a total of of $20,160$ points (288 longitudinal and 70 latitudinal) observed during 15 days, for a total of $302,400$ observations.

TOMS DATA

Scenario

- For the missing data: follow Jun and Stein (2008): local averaging (24 observations) for each local averaging.
- Likelihood estimation unfeasible: select a subgrid of 336 spatial points and all temporal observations, for a total of $5,040$ observations.
- Detrend the data using spatio-temporal splines
- Residuals as a realization from a zero mean space-time Gaussian random field.

TOMS DATA

Scenario

A. Two models based on the adapted Gneiting classes

B. Three models based on direct construction, hence valid with GC only.

C. A model based on the Gneiting class valid using CH and MP distances

TOMS DATA

Scenario

Distance	GC	CH	MP	GC	CH	MP	GC	CH	MP
Model		A.1			A.2			C.1	
c_S	742.7	743.9.3	734.2	681.3	672.2	733.3	-	450.8	417.5
c_T	2.54	2.54	2.17	1.76	1.67	1.71	-	1.56	1.29
β	1	1	0.95	1	1	1	-	0.95	0.89
σ^2	102.6	102.1	102.7	106.1	103.0	111.4	-	97.9	98.5
Nugget	9.81	9.82	6.35	6.30	5.60	4.97	-	12.26	5.72
Likelihood	-17233.8	-17234.0	-17296.5	-17257.1	-17258.5	-17317.3	-	-17156.3	-17223.9

Model		B.1			B.2			B.3	
τ	0.01	-	-	144.09	-	-	4.04	-	-
c_T	51.05	-	-	50.58	-	-	36.85	-	-
α	1.49	-	-	1.49	-	-	1.61	-	-
σ^2	85.8	-	-	86.0	-	-	89.9	-	-
Nugget	18.60	-	-	18.54	-	-	16.84	-	-
Likelihood	-17168.3	-	-	-17167.9	-	-	-17162.8	-	-