Basics and recent developments on spatio-temporal point processes

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Point process = random point field.

Spatio-temporal point process

- Involves temporal as well as spatial dispersion of points.
- Stochastic process governing the location and time of presence of points, so called *events*, where the number of such events is also random.

Spatio-temporal point pattern

Realization of a spatio-temporal point process, usually restricted to a spatio-temporal region $W_S \times W_T \subset \mathbb{R}^d \times \mathbb{R}$, $d \geq 1$ (in the following, $d = 2$).

It is described as a collection of pairs (s_i, t_i), $i = 1, \ldots, n$ where s_i and t_i are the location and time of occurrence associated with the ith event.

Basic questions

- Is the point pattern clustered/random/regular ?
- Is there any interaction between events ?

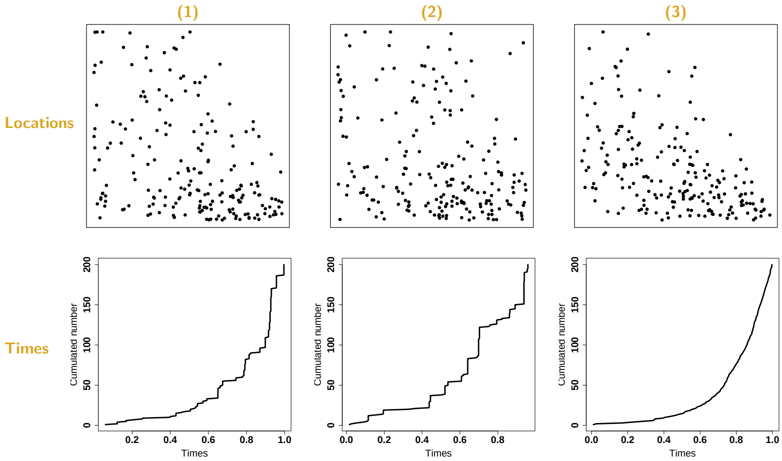
  $\Rightarrow$ **Analyzing spatio-temporal point process data :**
  Edith, Ottmar and Francisco

- Which model for the underlying point process ?
- How to fit its parameters ?

  $\Rightarrow$ **Modeling and inferring spatio-temporal models :**
  Thomas, Samuel

Motivation

3 realisations : not many differences at first sight



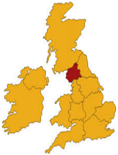
$\Rightarrow$ How to catch differences ?

Plotting the data : first step of any exploratory analysis

Illustrations (and analyses) from the R package stpp [1]

Data : UK 2001 foot-and-mouth disease

- Daily reports of confirmed cases :
  First case 19 February 2001 ; last confirmed : 30 September 2001
- 44 counties affected : more severely in Cumbria
  About 648 animal-holding farms (•) suffered cases over the 5153 (·)

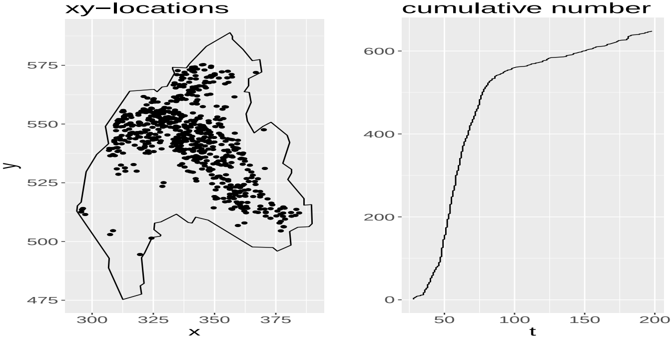


1. Gabriel, Rowlingson and Diggle (2013) *Journal of Statistical Software*, 53(2) :1–29.

Static plot (1)

Separate plots of locations s_i and times t_i, $i = 1, \ldots, n$

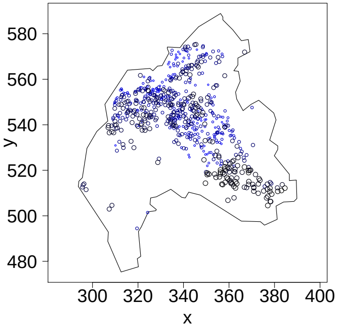
2001 FMD : Locations and cumulative distribution of times



Static plot (2)

Time treated as a quantitative mark attached to each location :

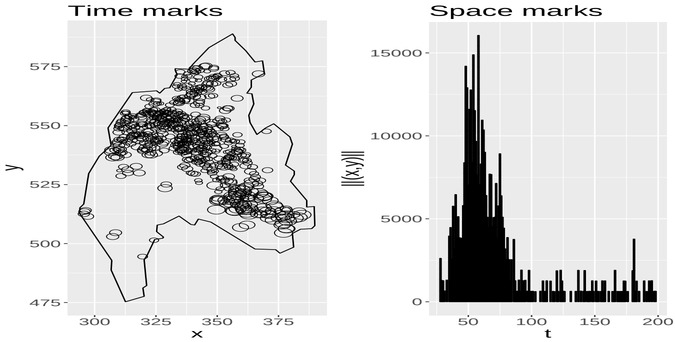
Locations are plotted with the size and/or colour of the plotting symbol determined by the value of the mark.



Former (○) to latter (○) cases

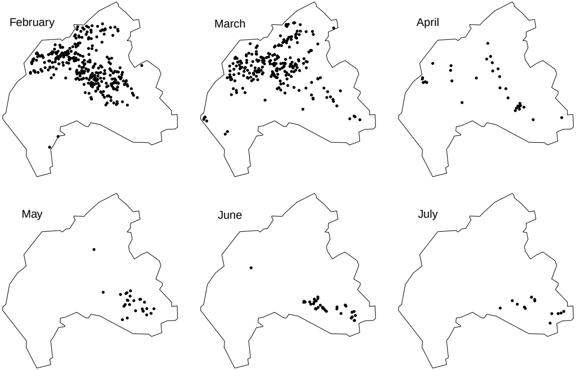
Static plot (3)

Times and locations as marks (see Francisco's talk)



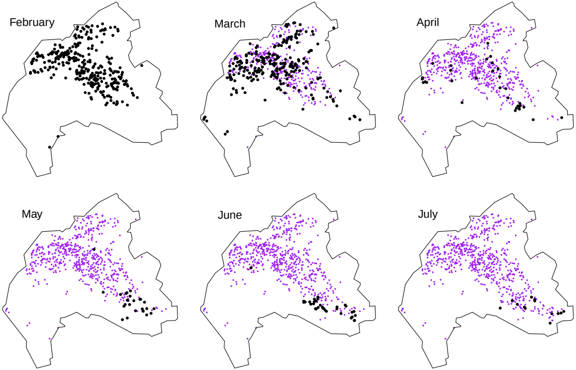
Static plot (4)

Plots of locations within time-intervals



Static plot (4)

...can also be superimposed over the previous events (•)



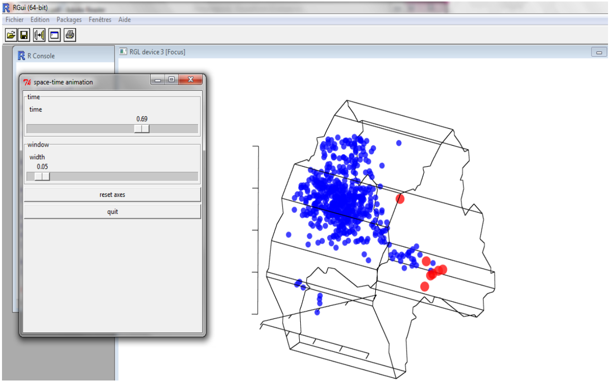
Dynamic plot (1)

2D : animation

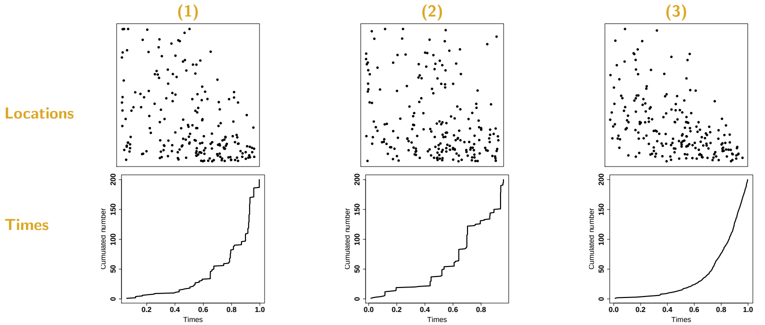
Events at time t and at time $< t$.

Dynamic plot (2)

3D



Plotting the data is not enough



⇒ Develop (suitable) statistical tools.

Second-order analysis

of spatio-temporal point process data

On spatio-temporal point processes

Notations

- Φ ou Φ_W : point process observed in $\in W = W_S \times W_T \subset \mathbb{R}^2 \times \mathbb{R}^+$
- $x_i = (s_i, t_i)$: the ith event,
- $\Phi(B) = \sum_{x \in \Phi} \mathbb{I}_W(x)$: number of points of Φ within the set B.

First-order moment

Intensity measure and intensity

The intensity measure Λ of Φ is defined by

$$\Lambda(B) = \mathbb{E}\left[\Phi(B)\right], \text{ for Borel sets } B.$$

Under some continuity conditions, $\Lambda(x)$ has density $\lambda(x)$, which is called *intensity function*.

$$\Lambda(B) = \int_B \lambda(x)\,\mathrm{d}x.$$

First-order moment

Intensity

Probability of one event within an elementary region :

$$\mathbb{P}[\text{ there is point of } \Phi \text{ in } ds_i \times dt_i \,] = \lambda(s_i, t_i)\, \mathrm{d}s_i\, \mathrm{d}t_i$$

where $ds_i \times dt_i$ is an elementary region centered at (s_i, t_i), with volume $\nu(ds_i \times dt_i)$.

$$\lambda(s_i, t_i) = \lim_{\nu(ds_i \times dt_i) \to 0} \frac{\mathbb{E}\left[\Phi(ds_i \times dt_i)\right]}{\nu(ds_i \times dt_i)}.$$

A very useful Theorem

An application of Fubini's Theorem :

Campbell Theorem

For any nonnegative measurable function $f(x)$,

$$\mathbb{E}\left[\sum_{x\in\Phi} f(x)\right] = \int f(x)\Lambda(\,\mathrm{d}x)$$

If Φ has an intensity function, then

$$\mathbb{E}\left[\sum_{x\in\Phi} f(x)\right] = \int f(x)\lambda(x)\,\mathrm{d}x.$$

Second-order moment

Second moment measure

The second order intensity measure $\mu^{(2)}$ of Φ is defined by

$$\mu^{(2)}(B_1 \times B_2) = \mathbb{E}\left[\Phi(B_1)\Phi(B_2)\right].$$

So we can write, $\mathbb{C}\mathrm{ov}(\Phi(B_1), \Phi(B_2)) = \mu^{(2)}(B_1 \times B_2) - \Lambda(B_1)\Lambda(B_2)$

The second factorial moment measure $\alpha^{(2)}$ of Φ is the intensity measure of all distinct points of Φ :

$$\alpha^{(2)}(B_1 \times B_2) = \mathbb{E}\left[\Phi(B_1)\Phi(B_2)\right] - \mathbb{E}\left[\Phi(B_1 \cap B_2)\right].$$

Applying Campbell's formula for the mean to $\Phi \times \Phi$ leads to

$$\mathbb{E}\left[\sum_{x\in\Phi}\sum_{x'\in\Phi} f(x,x')\right] = \iint f(x,x')\mu^{(2)}(\mathrm{d}x,\mathrm{d}x') \text{ and } \mathbb{E}\left[\sum_{x,x'\in\Phi}^{\neq} f(x,x')\right] = \iint f(x,x')\alpha^{(2)}(\mathrm{d}x,\mathrm{d}x').$$

Second-order moment

Second-order intensity function

The process Φ is said to have second moment density (or a second-order intensity function) λ_2 if

$$\alpha^{(2)}(B_1 \times B_2) = \int_{B_1} \int_{B_2} \lambda_2(x, x')\, \mathrm{d}x\, \mathrm{d}x'.$$

Probability of two events, each within an elementary region :

$$\mathbb{P}\left[\begin{array}{c} \text{one point of } \Phi \text{ in } ds_i \times dt_i \\ \text{and} \\ \text{one point of } \Phi \text{ in } ds_j \times dt_j \end{array}\right] = \lambda_2((s_i, t_i), (s_j, t_j))\, \mathrm{d}s_i\, \mathrm{d}t_i\, \mathrm{d}s_j\, \mathrm{d}t_j$$

$$\lambda_2((s_i, t_i), (s_j, t_j)) = \lim_{\nu(ds_i \times dt_i) \to 0, \nu(ds_j \times dt_j) \to 0} \frac{\mathbb{E}\left[\Phi(ds_i \times dt_i)\Phi(ds_j \times dt_j)\right]}{\nu(ds_i \times dt_i)\nu(ds_j \times dt_j)}$$

Second-order moment

Pair correlation function

Relationship between number of events in a pair of subregions

$$g((s_i, t_i), (s_j, t_j)) = \frac{\lambda_2((s_i, t_i), (s_j, t_j))}{\lambda(s_i, t_i)\lambda(s_j, t_j)}$$

Difficult to estimate $\Rightarrow$ relaxing assumptions.

Usual relaxing assumptions

- **First-order stationarity** : $\lambda(s_i, t_i) = \lambda$
- **First-order separability** : $\lambda(s_i, t_i) = \lambda_S(s_i)\lambda_T(t_i)$
- **Second-order stationarity** :

$$\lambda(s_i, t_i) = \lambda \text{ and } \lambda_2((s_i, t_i), (s_j, t_j)) = \lambda_2(s_i - s_j, t_i - t_j)$$

  If Φ is also isotropic,
  $\lambda_2((s_i, t_i), (s_j, t_j)) = \lambda_2(r, t)$, with $r = \|s_i - s_j\|$ and $t = |t_i - t_j|$.

- **Second-order separability** :

$$g((s_i, t_i), (s_j, t_j)) = g_S(s_i, s_j)g_T(t_i, t_j)$$

  If Φ is also isotropic, $g((s_i, t_i), (s_j, t_j)) = g_S(r)g_T(t)$.

Summary characteristics

Various summary characteristics have been proposed which describe particular features of Φ.

These are typically real number or functions based on inter-point (spatial and temporal) distances.

Their interpretation is a question of experience (and somewhat of an art ! [2])

2. Stoyan, Kendall & Mecke (1995) Stochastic geometry and its applications.

First-order characteristics : the intensity $\lambda(s, t)$

The **intensity** (= point density) gives a global information about Φ.

$\rightsquigarrow$ **Often associated with large scale inhomogeneity.**

It is of little value if alone.

The intensity influences the other summary characteristics.

Second-order characteristics : pcf and K-function

The **pair correlation function** is $g((s_i, t_i), (s_j, t_j)) = \dfrac{\lambda_2((s_i, t_i), (s_j, t_j))}{\lambda(s_i, t_i)\lambda(s_j, t_j)}$.

For a 2^{nd}-order stationary and isotropic point process :

$$g(r, t) = \frac{\lambda_2(r, t)}{\lambda^2}, \text{ with } r = \|s_i - s_j\| \text{ and } t = |t_i - t_j|$$

and the K-**function** is defined by

$$\lambda K(r, t) = \mathbb{E}\left[\text{ mean number of points within distances } r \text{ and } t \text{ from any point } \right].$$

$\rightsquigarrow$ **Often associated with short scale inhomogeneity.**

Second-order characteristics give information on many scales of distances.

The pcf does not contain more information than K, but is easier for interpretation (as non-cumulative).

Spatio-temporal inhomogeneous K-function [3]

Second-order intensity reweighted stationarity[2] (SOIRS) :

$$\lambda(s_i, t_i) \text{ non-constant and } \lambda_2((s_i, t_i), (s_j, t_j)) = \lambda_2(s_i - s_j, t_i - t_j)$$

For a SOIRS process, $r, t > 0$ and a compact B

$$K(r,t) = \frac{1}{\nu(B)} \mathbb{E}\left[\sum_{(s_i,t_i)\in\Phi\cap B} \sum_{(s_j,t_j)\in\Phi\setminus(s_i,t_i)} \frac{\mathbb{I}_{\{\|s_i - s_j\| \le r; |t_i - t_j| \le t\}}}{\lambda(s_i,t_i)\lambda(s_j,t_j)}\right]$$

If the process is also isotropic :

$$K(r,t) = \iint g(u,v)\mathbb{I}_{\{u\le r; v\le t\}}\,\mathrm{d}u\,\mathrm{d}v = 2\pi \int_{-t}^{t}\int_0^r g(u,v)u\,\mathrm{d}u\,\mathrm{d}v$$

Time-directional version :

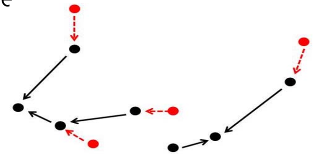
$$K(r,t) = 2\pi \int_{\mathbf{0}}^{t}\int_0^r g(u,v)u\,\mathrm{d}u\,\mathrm{d}v$$

3. Gabriel & Diggle (2009), *Statistica Neerlandica*, 63 :43–51.

Distance and contact distribution functions

Two distribution functions of the distance :

- from a point $\bullet = (s_o, t_o)$ of Φ
- from any point $\color{red}{\bullet} = (s^*, t^*)$ in W



Nearest neighbor distance :

$$G(r,t) = \mathbb{P}[d((s_o,t_o),\Phi\backslash\{(s_o,t_o)\}) \leq (r,t)] = 1 - \mathbb{E}^{!(s_o,t_o)}\left[\prod_{(x_i,t_i)\in\Phi}\left(1 - \frac{\bar{\lambda}\mathbb{I}_{\{\|s_i-s_o\|\leq r;|t_i-t_o|\leq t\}}}{\lambda(s_i,t_i)}\right)\right]$$

Empty space function :

$$H(r,t) = \mathbb{P}[d((s^*,t^*),\Phi) \leq (r,t)] = 1 - \mathbb{E}\left[\prod_{(x_i,t_i)\in\Phi}\left(1 - \frac{\bar{\lambda}\mathbb{I}_{\{\|s_i-s^*\|\leq r;|t_i-t^*|\leq t\}}}{\lambda(s_i,t_i)}\right)\right]$$

E.g : for clustered patterns, G gives information on the distances of the points within clusters and H describes the extent of empty space between clusters.

Spatio-temporal inhomogeneous J-function [4]

$$J(r,t) = \frac{1 - G(r,t)}{1 - H(r,t)},$$

with

- $G(r,t) = \mathbb{P}[d((s_o,t_o),\Phi\backslash\{(s_o,t_o)\}) \leq (r,t)]$ the nearest neighbor distance
- and $H(r,t) = \mathbb{P}[d((s^*,t^*),\Phi) \leq (r,t)]$ the empty space function :

For a SOIRS process,

$$J(r,t) - 1 \approx -\bar{\lambda}\left(K(r,t) - 2\pi r^2 t\right),$$

with $\bar{\lambda} = \inf_{(s,t)} \lambda(s,t)$.

4. Cronie & van Lieshout (2015) Scandinavian Journal of Statistics, 42(2) :562–579

Summary characteristics

Summary characteristics can be used for :

- Analyzing the spatio-temporal structure of a point pattern,

Statistic	Homogeneous Poisson process	Regular	Random	Clustered
$g(r,t)$	$g(r,t) = 1$	< 1	$= 1$	> 1
$K(r,t)$	$K(r,t) = 2\pi r^2 t$	$< 2\pi r^2 t$	$= 2\pi r^2 t$	$> 2\pi r^2 t$
$J(r,t) = \frac{1-G(r,t)}{1-H(r,t)}$	$G(r,t) = H(r,t)$	> 1	$= 1$	< 1

$\Rightarrow$ Deviation and pointwise envelop tests.

- Model fitting and estimation parameters.

Pointwise envelope tests based on 2^d-order moments

- Test of clustering/regularity

  H_0^c : "the pattern is a realisation of a Poisson process with intensity $\lambda(s,t)$."

  Under H_0^c, $g(r,t) = 1$ and $K(r,t) = 2\pi r^2 t$

  $\Rightarrow$ Confidence envelopes built from simulations of a $\mathcal{P}ois\,(\lambda(s,t))$.

- Test of interaction

  H_0^i : "the pattern is a realisation of a pair of independent spatial and temporal, second-order intensity reweighted stationary point processes."

  Under H_0^i, $K(r,t) \propto K_S(r)K_T(t)$ (second-order separability).

  $\Rightarrow$ Confidence envelopes built by random labelling the locations of events, holding their times fixed.

Deviation tests based on 2^d-order moments

Deviation measures :

- Integral deviation measure $\int_{t_{min}}^{t_{max}} \int_{r_{min}}^{r_{max}} \left(T(r,t) - T_{H_0}(r,t)\right)^q \, \mathrm{d}r \, \mathrm{d}t$
- Supremum deviation measure $\sup_{(r,t)} |T(r,t) - T_{H_0}(r,t)|$

where T can be the K, g, J, ...

$\Rightarrow$ Monte-Carlo tests with the deviation measure computed from

- the data, T_1.

- simulations under H_0, T_i, $i = 2, \ldots, N$.

T_{H_0} is often replaced by $\bar{T}$.

Usual models and simulations

Hypotheses testing is often based on Monte Carlo simulations.

Various spatio-temporal models are implemented in the R package `stpp`

Spatio-temporal models

⋆ **Independent processes**

- Inhomogeneous Poisson processes are widely used.
- The position of the clusters is fixed.
- There is no interaction between points.

⋆ **Dependent processes**

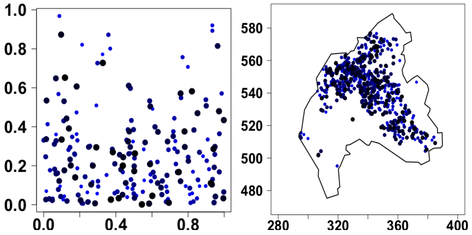
- There is interaction between points.
- Patterns follow different principles :
  - aggregation : Poisson cluster process, contagious processes, . . .
  - regularity : Inhibition process,
  - stochastic environment : Cox process.

Inhomogeneous Poisson process

It is the simplest non-stationary point process.

It is defined by the following postulates :

1. The number $\Phi(W_S \times W_T)$ of events within the region $W_S \times W_T$ follows a Poisson distribution with mean $\int_{W_S} \int_{W_T} \lambda(s,t)\, \mathrm{d}t\, \mathrm{d}s$.
2. Given $\Phi(W_S \times W_T) = n$, the n events in $W_S \times W_T$ form an independent random sample from the distribution on $W_S \times W_T$ with probability density function $f(s,t) = \lambda(s,t) / \int_{W_S} \int_{W_T} \lambda(s',t')\, \mathrm{d}t'\, \mathrm{d}s'$.



Cox process

Definition

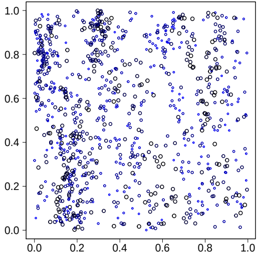
1. $\{\Lambda(s,t) : (s,t) \in W_S \times W_T\}$ is a non-negative-valued stochastic process.
2. Conditional on $\{\Lambda(s,t) = \lambda(s,t) : (s,t) \in W_S \times W_T\}$, the events form an inhomogeneous Poisson process with intensity $\lambda(s,t)$.

Log-Gaussian Cox process : $\Lambda(s,t) = \exp(Z(s,t))$ with $Z(s,t)$ a real-valued Gaussian field.

Covariance models $c(h,t)$:

- Separable, non-separable.
- Isotropic, anisotropic.

$g(r,t) = \exp(c(r,t))$.



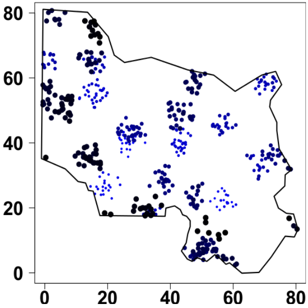
Poisson Cluster Process

Definition

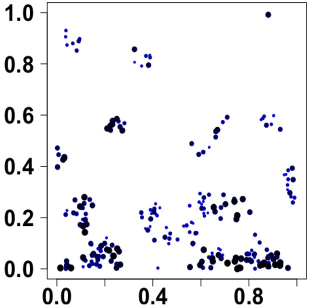
1. Parents form a Poisson process with intensity $\lambda_p(s,t)$.
2. The number of offspring per parent is a random variable N_c with mean m_c, realised independently for each parent.
3. The positions and times of the offspring relative to their parents are independently and identically distributed according to a trivariate probability density function $f(\cdot)$ on $\mathbb{R}^2 \times \mathbb{R}^+$.
4. The final process is composed of the superposition of the offspring only.

Poisson Cluster Process

Homogeneous parents distribution



Inhomogeneous parents distribution



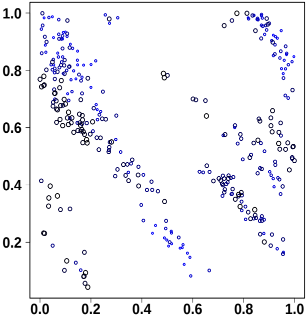
$$g(r,t) = 1 + \frac{\alpha}{8\pi\sigma^2\nu} \exp\left(-\frac{r^2}{4\sigma^2} - \alpha t\right),$$

$$K(r,t) = 2\pi r^2 t + \frac{1}{2\nu}\left(\exp(\alpha t) - \exp(-\alpha t)\right)\left(1 - \exp\left(-\frac{r^2}{4\sigma^2}\right)\right).$$

Anisotropic Poisson Cluster Process

Geometric anisotropy : $g(u,t) = g_0\left(\sqrt{u\Sigma^{-1}u'}, t\right)$,

where $u \in \mathbb{R}^2$ is a row vector with transpose u', Σ is a 2×2 symmetric positive definite matrix of the form $\Sigma = \omega^2 U_\theta \text{diag}(1, \zeta^2) U_\theta^t$ with $U_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$.



Interaction process

Inhibition process :

$\rightsquigarrow$ Make unlikely the occurrence of pairs of close events.

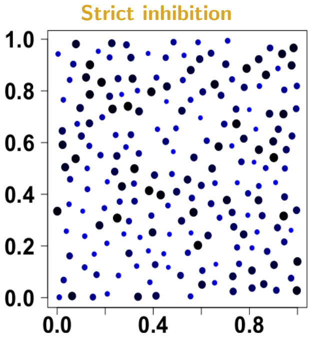
⋆ Simple sequential inhibition process

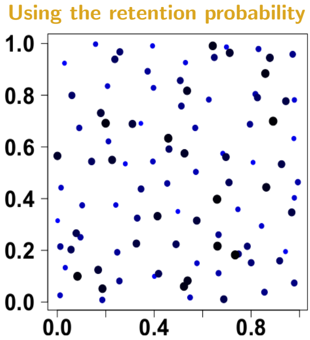
1. s_1 and t_1 are uniformly distributed in W_S and W_T respectively.
2. At the kth step of the algorithm, $k = 2, ..., m$,
   $s_k \sim \mathcal{U}\big[W_S \cap \{s : \|s - s_j\| \geq \delta_s, j = 1, \ldots, k-1\}\big]$
   $t_k \sim \mathcal{U}\big[W_T \cap \{t : |t - t_j| \geq \delta_t, j = 1, \ldots, k-1\}\big]$.

   δ_s, δ_t : minimum permissible spatial and temporal distances between events.

⋆ Larger class of inhibition :
Introduce in 2. the probability that a potential point (s, t) will be accepted as a point of the process according to the R most recent events.

Inhibition process





Interaction process

Contagious process :

⋆ Simple model :

1. s_1 and t_1 are uniformly distributed in W_S and W_T respectively.
2. At the kth step of the algorithm, given $\{(s_j, t_j), j = 1, \ldots, k-1\}$,
   $s_k \sim \mathcal{U}\big[W_S \cap \{s : \|s - s_{k-1}\| \leq \delta_s\}\big]$,
   $t_k \sim \mathcal{U}\big[W_T \cap \{t : |t - t_{k-1}| \leq \delta_t\}$,
   δ_s, δ_t : maximum permissible spatial and temporal distances between events.

⋆ Larger class of contagion :
Introduce in 2. the probability that a potential point (s, t) will be accepted as a point of the process according to the R most recent events.

Contagious process

Simple contagious model **Using the retention probability**

Infectious process

Infectious disease : can be contracted by a person without their having come into direct contact with an infected person
($\neq$ contagious disease : transmitted only by direct contact.)

Here, an *infectious process* is such that to each infected individual at a time t there corresponds an infection rate $h(t)$, which depends on

- a latent period α,
- the maximum infection rate β,
- the infection period γ.

1. Choose the location s_1 and time t_1 of the first event.
2. Given $\{(s_j, t_j), j = 1, \ldots, k-1\}$,

   s_k is either symmetrically distributed around s_{k-1} or is a point in a Poisson($\lambda(s)$),

   t_k is either uniformly or exponentially distributed around t_{k-1}.

   A potential point is accepted with probability $p_k = f(h(t|t_{k-1}, \alpha, \beta, \gamma))$.

Infectious process

Spatio-temporal modelling

- Empirical models : e.g. Cox process

  Describe the point pattern without pointing to any particular underlying mechanism

- Mechanistic models : e.g. Interaction process

  Parameters make the link with generating process.

  Properties of the process are specified conditionally on its realization up to the current time.

  Conditional intensity function : $\lambda(s, t|\mathcal{H}_t)$,

  where $\mathcal{H}_t$ is the history of the process up to time t

See Diggle's 2013 book, Gabriel *et al.* (2013) and Gabriel (2016) papers, and **Thomas' talk**.

Inference for spatio-temporal models

Method of moments

But in spatial statistics :

- moments often unreachable $\Rightarrow$ (heavy) simulations.
- no solution or not unique.

Minimum contrast method

$$\hat{\theta} = \text{argmin}_\theta \int_a^b |T_\theta(x) - \mathbb{E}[T_\theta(x)]|^\beta \, dx$$

Usually : $\beta = 2,\ T(x) = K(r,t)$.

Likelihood-based method

But the likelihood is unreachable, except for Poisson and Gibbs processes, or approximately for Cox processes.

Estimation of the second moment measures

Estimating 2^d-order moments

Non-parametric estimation [5]

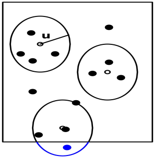
$$\widehat{K}(r,t) = \sum_{i=1}^{n}\sum_{j\neq i}\frac{1}{w_{ij}}\frac{\mathbb{I}_{\{\|s_i-s_j\|\leq r\ ;\ |t_i-t_j|\leq t\}}}{\lambda(s_i,t_i)\lambda(s_j,t_j)}$$

$$\widehat{g}(r,t) = \frac{1}{4\pi r}\sum_{i=1}^{n}\sum_{j\neq i}\frac{1}{w_{ij}}\frac{k_s(r-\|s_i-s_j\|)k_t(t-|t_i-t_j|)}{\lambda(s_i,t_i)\lambda(s_j,t_j)},$$

where w_{ij} is an edge correction factor,

$k_s(\cdot)$, $k_t(\cdot)$ are kernel functions (usually box or epanechnikov kernels).

Illustration of edge effects in 2D



5. Gabriel & Diggle (2009) ; Gabriel (2014).

Estimating 2^d-order moments

Problems :

(1) Edge effects have to be corrected.

(2) $\lambda(s, t)$ is not known and have to be estimated.

$\Rightarrow$ Which method can be used to correct edge effects ?
to estimate $\lambda(s, t)$?
What are their influence on the performance of $\widehat{K}$, $\widehat{g}$?

} see Gabriel (2014)

Edge correction methods [6]

- Isotropic method : $w_{ij} = |W_S \times W_T| w_{ij}^{(s)} w_{ij}^{(t)}$

  where $w_{ij}^{(s)}$ is the Ripley's method : proportion of the circumference of a circle centred at the location s_i with radius $\|s_i - s_j\|$ lying in W_S,

  $w_{ij}^{(t)} = 1$ if both ends of the interval of length $2|t_i - t_j|$ centred at t_i lie within W_T and $w_{ij}^{(t)} = 1/2$ otherwise.

- Border method : $w_{ij} = \dfrac{\sum_{j=1}^{n} \mathbb{I}_{\{d(s_j,W_S)>u \ ; \ d(t_j,W_T)>v\}} / \lambda(s_j, t_j)}{\mathbb{I}_{\{d(s_i,W_S)>u \ ; \ d(t_i,W_T)>v\}}}$

  where $d(s_i, W_S)$ (resp. $d(t_i, W_T)$) denotes the distance between s_i (resp. t_i) and the boundary of W_S (resp. W_T).

- Translation method : $w_{ij} = |W_S \cap W_{S_{s_i - s_j}}| \times |W_T \cap W_{T_{t_i - t_j}}|$,

  where $W_{S_{s_i - s_j}}$ and $W_{T_{t_i - t_j}}$ are the translated spatial and temporal regions.

6. Gabriel (2014) Methodology and Computing in Applied Probability, 16(2) :411–431.

Performances wrt edge correction methods

Performance of $\widehat{K}$ and $\widehat{g}$ for (in)homogeneous and/or (an)isotropic point patterns

Highest relative variance efficiency of $\int_0^r \int_0^t (\widehat{T}(u,v) - T(u,v))^2 \,\mathrm{d}u\,\mathrm{d}v$ with $T = K$ or $T = g$ obtained for

	$\widehat{K}$	$\widehat{g}$
Homogeneous Poisson process	Border	Border
Inhomogeneous Poisson process	Border	Translation
Isotropic clustered process	Translation	Translation
Anisotropic clustered process	-	Translation
(Weakly) Inhomogeneous clustered process	Translation	Translation
(Strongly) Inhomogeneous clustered process	Border	Border

Analyzing anisotropic point patterns [7]

For a SOIRS process Φ,

$$\widehat{K}(r, t, \theta) = \sum_{i=1}^{n} \sum_{j \neq i} \frac{1}{w_{ij}} \frac{\mathbb{I}_{\{\|s_i - s_j\| \leq r \ ; \ |t_i - t_j| \leq t ; \mathrm{A}(s_i, s_j) \leq \theta\}}}{\lambda(s_i, t_i)\lambda(s_j, t_j)}$$

with $\mathrm{A}(s_i, s_j)$ the least angle between the x-axis and the line defined by s_i and s_j.

If Φ is stationary and isotropic,

$$K(r, t) = K(r, t, 2\pi) = 2K(r, t, \pi).$$

7. Comas, Rodriguez-Cortes, Mateu (2015)

Analyzing anisotropic point patterns

Testing anisotropic effects [8]

For a SOIRS and isotropic process,

$$O(\theta) = \int_0^{\theta} \int_{r_{min}}^{r_{max}} \int_{t_{min}}^{t_{max}} \mathrm{d}K(r,t,\varphi) \ / \ \int_0^{\pi} \int_{r_{min}}^{r_{max}} \int_{t_{min}}^{t_{max}} \mathrm{d}K(r,t,\varphi)$$

is uniform on $[0,\pi)$.

Kolmogorov test based on

$$D = \sup_{\theta \in [0,\pi)} |\widehat{O}(\theta) - \theta/\varphi|.$$

8. Comas, Conde, Mateu (2016)

Analyzing marked point patterns

A marked point process is a point process with characteristics attached to each point.

A spatio-temporal marked point process is random sequence $\{[x_i, m_i]\}$ from which

- the points $x_i = (s_i, t_i)$ together constitute a point process in $\mathbb{R}^2 \times \mathbb{R}^+$,
- the m_i are the marks belonging to a given space of marks $\mathbb{M}$.

See Ottmar and Francisco talks !

Estimation and prediction of the intensity

Estimating 1^{st}-order moments

Intensity estimation (see e.g. Illian *et al.*, 2008)

Kernel estimation

- Useful when there is no covariates.
- The first-order separability is assumed.
- $\widehat{\lambda}(s,t) = \frac{1}{n}\widehat{\lambda}_S(s)\widehat{\lambda}_T(t)$,

with $\widehat{\lambda}_S(s) = \sum_{i=1}^{n} \frac{k_h(s-s_i)}{c_{W_S}(s_i)}$ and k_h a bivariate kernel with bandwidth h and $c_{W_S}(s_i) = \int_{W_S} k_h(s-s_i)\,\mathrm{d}s$ is an edge-correction factor to guarantee that $\int_{W_S} \widehat{\lambda}(s)\,\mathrm{d}s = n$.

Parametric estimation (see Thomas' talk)

Estimating 1^{st}-order moments

Performance of $\widehat{g}$ and $\widehat{K}$ may be severely altered by $\widehat{\lambda}$:

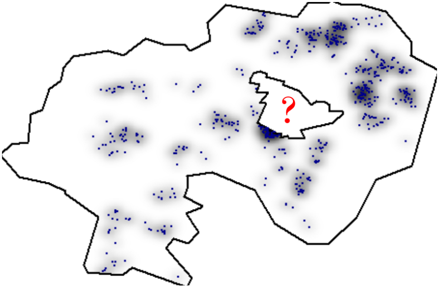
- $\rightsquigarrow$ parametric estimation : overparametrisation or overfitting,
- $\rightsquigarrow$ kernel estimation : incapacity of distinguish 1^{st} and 2^{d}-order effects from a single realisation of the point process.

  $\Rightarrow$ assumption : 1^{st}-order effects operate at larger scale than the 2^{d}-order effects.

  care needed when partitioning spatio-temporal patterns into 1^{st} and 2^{d}-order effects ; the knowledge about the environment is crucial.

Predicting the 1^{st}-order intensity [9]

How to get the intensity outside the observation window ?



For a SOIRS point process Φ observed in W and $x_o \notin W$,

$$\widehat{\lambda}(x_o|\Phi_W) = \int \omega(x;x_o) \sum_{y\in\Phi_W} \delta(x-y)\,\mathrm{d}x = \sum_{x\in\Phi_W} \omega(x;x_o)$$

is the Best Linear Unbiased Predictor of $\lambda(x_o|\Phi_W)$, with δ the dirac delta function.

9. Gabriel, Coville & Chadœuf (2017) Spatial Statistics + work in progress with F. Rodriguez, J. Mateu

Predicting the 1^{st}-order intensity

The weight function $\omega(x;x_o)$ satisfies the constraint

$$\int_W \lambda(x)\omega(x;x_o)\,\mathrm{d}x = \lambda(x_o)$$

and is solution of the Fredholm equation of the second kind :

$$\lambda(x)\omega(x;x_o) + \int_W \lambda(y)\omega(y;x_o)k(x,y)\,\mathrm{d}y = f(x;x_o),$$

with kernel

$$k(x,y) = \lambda(x)\left(g(x-y) - \frac{1}{\int_W \lambda(z)\,\mathrm{d}z}\int_W \lambda(z)g(z-y)\,\mathrm{d}z\right)$$

and source term

$$f(x;x_o) = \lambda(x)\lambda(x_o)\left(\frac{1}{\int_W \lambda(z)\,\mathrm{d}z} + g(x-x_o) - \frac{1}{\int_W \lambda(z)\,\mathrm{d}z}\int_W g(z-x_o)\,\mathrm{d}z\right).$$

Approximated solution

Finite element approach

The Galerkin method, with $\mathcal{T}_h$ a mesh partitioning W and V_h an approximation space, plugged into a weak formulation of the Fredholm equation, leads to :

$$\sum_{j=1}^{N} w_j \int_W \left(\varphi_i(x)\varphi_j(x) + \int_W \int_W k(x,y)\varphi_j(y)\varphi_i(x)\,\mathrm{d}y \right) = \int_W f(x;x_o)\varphi_i(x)\,\mathrm{d}x,$$

with $\omega(x;x_o) \approx \sum_{i=1}^{N} w_i\varphi_i(x)$, $N = \dim V_h$ and $\{\varphi_i\}_{i=1,\dots,N}$ a basis of V_h.

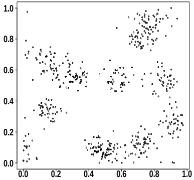
Illustrative results in 2D (1)

Simulation of a Thomas process within $[0,1] \times [0,1]$

Parents : $\mathcal{P}ois(\mu)$, $\mu = 50$

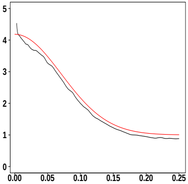
Offspring : $\mathcal{P}ois(\kappa)$, $\kappa = 10$, normally distributed, with $\sigma = 0.05$

Point process realization



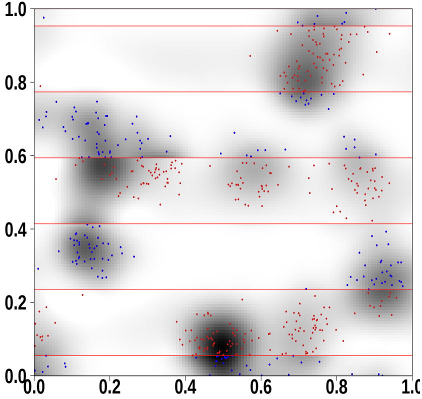
$\lambda = \kappa\mu = 500$

Pair correlation function



$g(r) = 1 + \frac{1}{4\pi\kappa\sigma^2} \exp\left(-\frac{r^2}{4\sigma^2}\right)$

Prediction within W_{unobs}

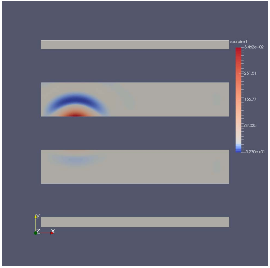


$\{\color{blue}\bullet\}$: Φ_W ; $\{\color{red}\bullet\}$: $\Phi_{W_{unobs}}$

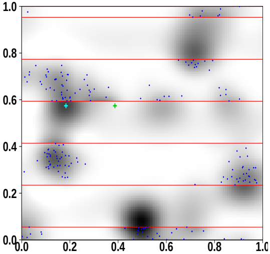
Illustrative results in 2D (1)

Weight function $\omega(\cdot; x_o)$

$x_o = (0.18, 0.57)$

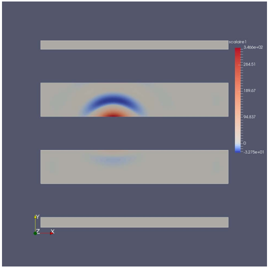


Prediction in W



$\{\bullet\}$: $\Phi_{W_{obs}}$

$x_o = (0.38, 0.57)$



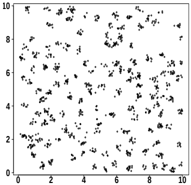
Illustrative results in 2D (2)

Simulation of a cluster process within $[0, 10] \times [0, 10]$

Parents : hardcore process with interaction radius 0.5

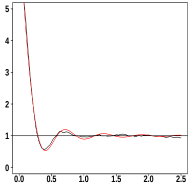
Offspring : normally distributed, with $\sigma = 0.1$

Point process realization



$\hat{\lambda} = 12.58$

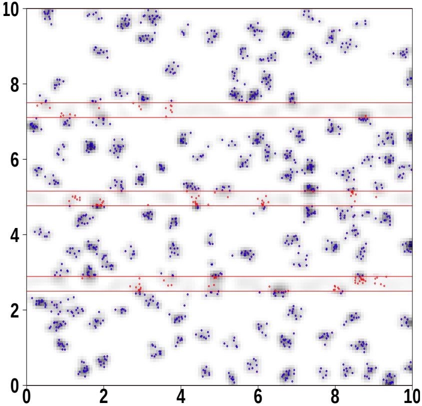
Pair correlation function



$$g(r) = 1 + \alpha \frac{\delta}{r} \exp\left(-\left(\frac{r}{\delta}\right)^{\beta}\right) \sin\left(\frac{r}{\delta}\right)$$

$\hat{\alpha} = 11.65\,;\ \hat{\beta} = 0.35\,;\ \hat{\delta} = 1.25$

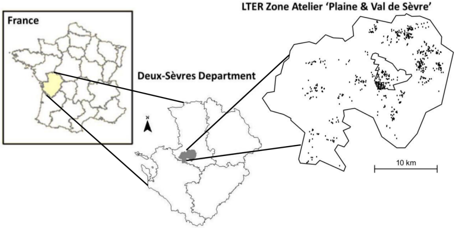
Prediction within W_{unobs}

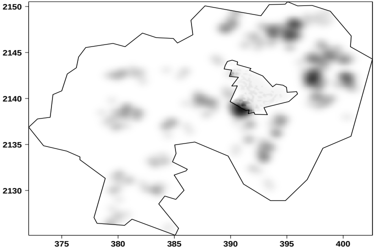


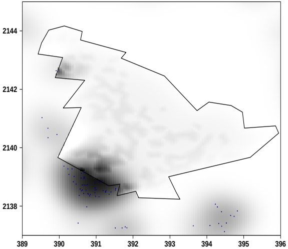
$\{\bullet\}$: Φ_W ; $\{\bullet\}$: $\Phi_{W_{unobs}}$

Application : in 2D

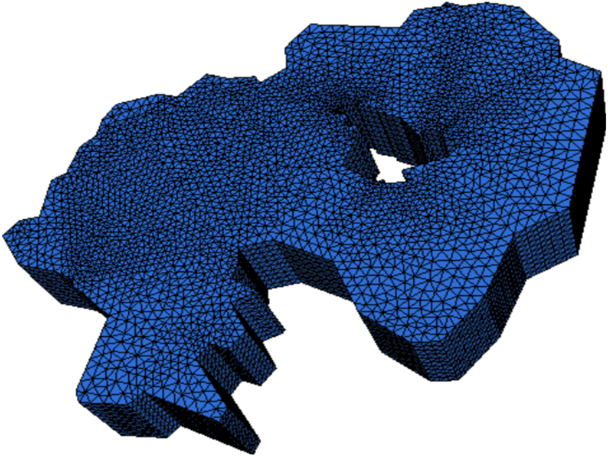
Montagu's Harriers nest locations







Application : in 3D



. . .work in progress

References

- O. Cronie and M-C. van Lieshout (2015) A J -function for inhomogeneous spatio-temporal point processes. *Scandinavian Journal of Statistics*, 42(2), 562–579.
- P.J. Diggle (2013). *Statistical analysis of spatial and spatio-temporal point patterns*, Third Edition. Chapman and Hall.
- E. Gabriel and P.J. Diggle (2009). Second-order analysis of inhomogeneous spatio-temporal point process data. *Statistica Neerlandica*, 63 :43–51.
- E. Gabriel, B. Rowlingson and P.J. Diggle (2013). STPP : Plotting, simulating and analysing spatio-temporal point patterns. *Journal of Statistical Software*, 53(2) :1–29.
- E. Gabriel (2014) Estimating second-order characteristics of inhomogeneous spatio-temporal point processes : influence of edge correction methods and intensity estimates. *Methodology and Computing in Applied Probability*, 16(2), 411–431.
- E. Gabriel (2016) Spatio-temporal point pattern analysis and modelling. *Encyclopedia of GIS*, 2nd Edition.
- E. Gabriel, J. Coville, J. Chadœuf (2017) Estimating the intensity function of spatial point processes outside the observation window. Spatial Statistics (to appear).
- J. Illian, A. Penttinen, H. Stoyan and D. Stoyan (2008) *Statistical Analysis and Modelling of Spatial Point Patterns*. John Wiley & Sons, London.
- D. Stoyan, D., F. Rodríguez-Cortés, J. Mateu,and W. Gille (2017). Mark variograms for spatio-temporal point processes, *Spatial Statistics*, 20 :125–147.